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TAYLOR SERIES CALCULUS FOR RING OBJECTS OF LINE TYPE

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We attempt here to present a foundation of a kind of Differential Algebra, where the differentiation process is not an added *structure*, but something which stems from a *property* of the ring object considered. Ring objects of this kind ("rings of line type") are not present in the category of sets, but occur in some of the toposes of algebraic geometry, as well as in the category of formal schemes.

There are two basic ideas. The first is an idea of Lawvere from 1967 that there should be an object D of infinitesimals such that the "function space object" M^D is the tangent bundle of M (for any, or for many, M). D must be a certain subobject of the ring object A in question. The second idea is that the tangent bundle A^D of A itself in a canonical way should be isomorphic to $A \times A$. There will in fact be a canonical map $\alpha : A \times A \to A^D$, and invertibility of this α is the only property of A we need to make a fair amount of differential calculus work.

The paper is a sequel to [1], in which models for the axiom were presented, and Leibniz- and chain-rule proved for differentiation of functions $A \to A$ ("in one variable"). We shall need these rules here also, but apart from that, we do not presuppose [1].

The consideration of the D_k's in Section 2, which is crucial to the proof of Taylor's Theorem, was suggested to me by Gavin Wraith, who also gave the correct version of the role of the Jacobians, and its proof. Also, he considered several years ago the object D in the category of formal schemes, and showed how much differential calculus stems from it, [5]. I want to thank him for many fruitful discussions on the present subject.

The plan of the paper is as follows. First we give the several-variable first order calculus. Because of the simplicity of the axiomatics, it applies not only to the category $\mathscr{E}$ in question, but also to each $\mathscr{E}/N$; this allows us to interpret partial differentiation in $\mathscr{E}$ as a special case of differentiation, namely differentiation in $\mathscr{E}/N$ (for suitable N); see Section 4.

Then we use the D_k's and a characteristic-zero assumption, to relate maps in the category with their Taylor series. The outcome of this is that certain maps in the category in question can be identified with formal power series; these maps are precisely those that are interesting from a local-geometric point of view. In this sense, our theory leads back to classical power-series algebra (inverse function theorem etc.): this is summed up in Theorem 5.3.

Finally, a word about a notation method. Commutativity of a diagram like, say,

$$\begin{array}{ccc} & & C\times C \\ & \overset{\langle f_1,f_2\rangle}{\nearrow} & \downarrow + \\ A\times B & \xrightarrow[g]{} & C \end{array} \tag{0.1}$$

is expressed by saying

$$f_1(\boldsymbol{a})+f_2(\boldsymbol{b})=g(\boldsymbol{a},\boldsymbol{b}) \qquad \forall \boldsymbol{a}\in A, \boldsymbol{b}\in B. \tag{0.2}$$

The reader may translate such an equation back into a diagram, or alternatively, interpret $\boldsymbol{a}$ and $\boldsymbol{b}$ as "elements" of A and B, respectively, in the sense: $\boldsymbol{a}$ and $\boldsymbol{b}$ are maps $X\to A$ and $X\to B$, where X is an unspecified object of the category. Similarly, we use such equations as descriptions of constructions of maps; e.g. (0.2) may be used as a definition of the map g in (0.1) in terms of the maps f_1, f_2 and $+$.

1. Jacobians and differentials

Let $\mathscr{E}$ be a category with finite inverse limits, and A a commutative ring object in $\mathscr{E}$. We let, as in [1], D denote the equalizer of the two maps $A\to A$ given by "squaring" and "constant 0". (So $a:X\to A$ factors through D if and only if $a^2=0$ in the ring $\hom(X,A)$).

We assume that the exponential object A^D exists, and we consider the map $\alpha: A\times A\to A^D$ whose exponential adjoint $\check\alpha: A\times A\times D\to A$ is given by the description

$$\check\alpha(a_0,a_1,d)=a_0+(a_1\cdot d).$$

As in [1], we say that A is of *line type* if α is invertible; in the sequel, we assume that this is the case.

Let M be an A-module object. We say that M is *Euclidean* if the map

$$\alpha_M: M\times M\to M^D$$

whose exponential adjoint

$$\check\alpha_M: M\times M\times D\to M$$

has the description

$$\check\alpha_M(\boldsymbol{m}_0,\boldsymbol{m}_1,d)=\boldsymbol{m}_0+(\boldsymbol{m}_1\cdot d)$$

is invertible; by assumption, A, and thus also $A^n = A\times\cdots\times A$ (n times), is Euclidean. (There do exist examples of A-module objects which are not Euclidean.) For a Euclidean object M, we denote by γ or γ_M the map

$$M^D \xrightarrow{\alpha^{-1}} M\times M \xrightarrow{\text{proj}_1} M$$

(where proj_1 is projection onto the *second* factor;—projection to the first factor is denoted proj_0).

We can equip $A\times A$ with a "ring-of-dual-numbers" ring structure: addition is coordinatewise, and multiplication

$$(A\times A)\times(A\times A)\to A\times A$$

is given by

$$\langle\langle a_0, a_1\rangle, \langle a_0', a_1'\rangle\rangle \mapsto \langle a_0\cdot a_0', a_0\cdot a_1' + a_1\cdot a_0'\rangle.$$

Likewise, the A-module $M\times M$ can be given a module structure over this ring $A\times A$, namely with multiplication

$$\langle\langle a_0, a_1\rangle, \langle m_0, m_1\rangle\rangle \mapsto \langle a_0\cdot m_0, a_0\cdot m_1 + a_1\cdot m_0\rangle.$$

We denote $A\times A$ and $M\times M$ by $A[\varepsilon]$ and $M[\varepsilon]$, respectively when we want to emphasize these structures. Also, since the (partially defined) functor $(\ )^D$ preserves products, A^D inherits a ring structure from A, and M^D inherits an A^D-module structure; these structures may be called the *diagonally induced* structures. With the structures thus described, we have:

Proposition 1.1. *The map $\alpha: A\times A\to A^D$ is a ring isomorphism; the map*

$$(\alpha, \alpha_M): (A\times A, M\times M)\to(A^D, M^D)$$

is a ring-module homomorphism (and thus an isomorphism in case M is Euclidean).

Proof. The first statement is proved in [1]; the proof of the second is similar.

We have clearly that $0: \mathbf{1}\to A$ factors through D. The map $\mathbf{1}\to D$ thus resulting gives, for any object N for which N^D exists, rise to a map

$$N^D \longrightarrow N^{\mathbf{1}} \cong N, \tag{1.1}$$

"the tangent bundle of N" (as in Lawvere [3], or Wraith [6]). A cross section $X: N\to N^D$ of p will be called a *tangent vector field* of N.

Suppose X is a tangent vector field and that $f: N\to M$ is any map into a Euclidean object M. Then by the *derivative* of f along X we understand the composite

$$N \xrightarrow{X} N^D \xrightarrow{f^D} M^D \xrightarrow{\gamma_M} M$$

denoted $D_X f$. (Again we utilize the (partial) functorality of $(\ )^D$). Now the set $\hom(N, M)$ carries a canonical structure of $\hom(N, A)$-module.

Proposition 1.2. *The pair of maps*

$$D_X : \hom(N, M) \to \hom(N, M); \qquad D_X : \hom(N, A) \to \hom(N, A)$$

is a ring-module derivation, i.e., it is additive, and

$$D_X(\varphi \cdot f) = D_X\varphi \cdot f + \varphi \cdot D_X f$$

for $f: N \to M$ and $\varphi: N \to A$. Further, $D_X f = 0$ iff f is constant (i.e. factors through $\mathbb{1}$).

Proof. We first note that if M is an A-module object for which M^D exists, then we have a commutative triangle

$$\begin{array}{ccc} M \times M & \xrightarrow{\alpha_M} & M^D \\ & {\scriptstyle \text{proj}_0} \searrow \quad \swarrow {\scriptstyle p} & \\ & M & \end{array} \tag{1.2}$$

where $p: M^D \to M$ is as in (1.1). (Compare Proposition 4 of [1]). Also, the $A[\varepsilon]$ and $M[\varepsilon]$ structures on $A \times A$ and $M \times M$, respectively, clearly makes the pair $\text{proj}_1: M \times M \to M$, $\text{proj}_1: A \times A \to A$ into a ring-module derivation with respect to the pair $\text{proj}_0: M \times M \to M$, $\text{proj}_0: A \times A \to A$. From Proposition 1.1 and (1.2) we then deduce that $\gamma_M: M^D \to M$, $\gamma: A^D \to A$ is a ring-module derivation with respect to $p: M^D \to M$, $p: A^D \to A$. But if X is a tangent vector field, $X: N \to N^D$, as above, and $f: N \to M$, then

$$p \circ f^D \circ X = f,$$

as is easily seen. The first part of the proposition then easily follows from the fact that $(f+g)^D = f^D + g^D$ and $(\varphi \cdot f)^D = \varphi^D \cdot f^D$ for the diagonally induced structures on M^D and A^D. The second part is proved by comparing the two relevant maps into M^D by by passage to exponential adjoints.

In case where $N = M = A$, and where X is the canonical vector field $\hat{+} : A \to A^D$ (given as exponential adjoint of $+ : A \times D \to A$), $D_X f$ is just the f' as considered in [1].

If U is any object and $X : U \to U^D$ a vector field, then we can use (1.2) to get an alternative description of $D_X f$ for an $f: U \to M$ into a Euclidean M; namely we have the existence of a commutative square

$$\begin{array}{ccc} U & \xrightarrow{\langle f, D_X f \rangle} & M \times M \\ {\scriptstyle X} \downarrow & & \downarrow {\scriptstyle \alpha_M} \\ U^D & \xrightarrow[f^D]{} & M^D \end{array} \tag{1.3}$$

(Proof. Replace α_M by α_M^{-1} and look at the two projections $M \times M \to M$).

There is another kind of "differentiation' which has good functorial properties: formation of differentials, df. To define this generally enough, we introduce the notion of sub-euclidean object (which will serve as a substitute for the notion of "open subset of Euclidean space"): Let N be a Euclidean object, and $u: U \rightarrowtail N$ a subobject. We say that it is *subeuclidean* provided U^D exists, and there exists a map α_U (necessarily unique) filling out the diagram

$$\begin{array}{ccc} U\times N & \xrightarrow{\alpha_U} & U^D \\ {\scriptstyle U\times N}\downarrow & & \downarrow{\scriptstyle U^D} \\ N\times N & \xrightarrow[\alpha_N]{} & N^D \end{array} \tag{1.4}$$

in a commutative way (dotted arrow α_U).

Now let $U \rightarrowtail N$ be subeuclidean, and $f: U \to M$ any map into a Euclidean M. By the *differential* df of f we understand the composite

$$df = U\times N \xrightarrow{\alpha_U} U^D \xrightarrow{f^D} N^D \xrightarrow{\gamma_N} N.$$

When using elementwise notation, we shall write $df_a(b)$ rather than $df(a, b)$.

The differential can be described alternatively, using (1.2): it sits in a commutative square (which should be compared to (1.3)):

$$\begin{array}{ccc} U\times N & \xrightarrow{\langle f\circ \text{proj}_0,\, df\rangle} & M\times M \\ {\scriptstyle\alpha_U}\downarrow & & \downarrow{\scriptstyle\alpha_M} \\ U^D & \xrightarrow[f^D]{} & M^D \end{array} \tag{1.5}$$

(Proof: Replace α_M by α_M^{-1} and look at the two projections $M\times M \to M$).

Now $(\ )^D$ is a functor on the full subcategory of those objects X for which X^D exists. These good functorial properties are then reflected via (1.5) into good functorial properties of the differential-formation:

Suppose we have the situation

$$\begin{array}{ccc} N & M & L \\ \uparrow & \nearrow_f\ \uparrow & \nearrow_g \\ U & V & \end{array}$$

with U and V subeuclidean in N and M, and L Euclidean, and suppose that f factors through V, $f: U \to V$. Then we have

Proposition 1.3. (Chain rule) *We have*

$$d(g\circ f)_a(b) = dg_{f(a)}(df_a(b)), \qquad \forall a\in U \quad b\in N.$$

(Properly speaking, this expresses commutativity of a certain diagram $U\times N \to L$, by the convention explained in the introduction).

Proof. This is a straightforward diagram chase, using $(g\circ f)^D = g^D \circ f^D$ and (1.5).

The existence of the map $\alpha_U : U\times N \to U^D$ (as in (1.4)) for a subeuclidean object $u : U \rightarrowtail N$ implies, by passing to exponential adjoints, the existence of a map $\check{\alpha}_U$ making

$$\begin{array}{ccc} U\times N\times D & \xrightarrow{\check{\alpha}_U} & U \\ {\scriptstyle U\times N\times D}\big\downarrow & & \big\downarrow{\scriptstyle u} \\ N\times N\times D & \xrightarrow[\check{\alpha}_N]{} & N \end{array}$$

commute, which we may express by saying "for all $\boldsymbol{b}\in N$, $\boldsymbol{a}\in U$, $\boldsymbol{a}+d\cdot\boldsymbol{b}\in U$ provided $d^2=0$" (thinking of d as an infinitesimal, $d\cdot\boldsymbol{b}$ is an infinitesimal vector, so subeuclidean objects have the property that they are ***stable under addition of infinitesimal vectors*** of this kind).

In order to proceed further, we have to describe differentials by means of Jacobi matrices. This cannot be done for arbitrary Euclidean objects, but it can be done for the A^n and their subeuclidean objects ("coordinate neighbourhoods").

Let $u : U \rightarrowtail A^n$ be a subeuclidean object of coordinate n-space. We have n distinguished tangent vector fields on U

$$\frac{\partial}{\partial x_i} : U \to U^D, \qquad i=1,\cdots,n,$$

$\partial/\partial x_i$ being the exponential adjoint of the map $U\times D\to U$ with description

$$\langle u_1,\cdots,u_n,d\rangle \to \langle u_1,\cdots,u_i+d,\cdots,u_n\rangle$$

or alternatively

$$\langle \boldsymbol{a}, d\rangle \xmapsto{\left(\frac{\partial}{\partial x_i}\right)^{\vee}} \boldsymbol{a}+d\cdot\boldsymbol{e}_i,$$

$\boldsymbol{e}_i$ denoting the i'th canonical basis vector $\mathbb{1}\to A^n$; $\boldsymbol{a}+d\cdot\boldsymbol{e}_i$ is in U by the above mentioned stability property of subeuclidean objects.

For $f:U\to M$ a map from a coordinate neighbourhood in A^n into a Euclidean object M, we denote by $\partial f/\partial x_i$ or $D_i f$ the derivative of f along the tangent vector field $\partial/\partial x_i$.

For $X=\partial/\partial x_i$, we now take the commutative diagram (1.3) and pass to exponential adjoints; this yields the diagram

$$\begin{array}{ccc} U\times D & \xrightarrow{\left\langle f,\frac{\partial f}{\partial x_i}\right\rangle\times D} & M\times M\times D \\ {\scriptstyle \left(\frac{\partial}{\partial x_i}\right)^{\vee}}\big\downarrow & & \big\downarrow{\scriptstyle \check{\alpha}_M} \\ U & \xrightarrow[f]{} & M \end{array}$$

whose commutativity is expressed by the equation

$$f(a+d\cdot e_i)=f(a)+d\cdot\frac{\partial f}{\partial x_i}(a) \tag{1.6}$$

(for $a\in U\subseteq A^n$, $b\in A^n$, $d\in D$). (This of course is a generalization of the "First Taylor Lemma", Proposition 6 of [1] to the many variable case; we shall later see that it can also be viewed as a *special* case of the 1-variable First Taylor Lemma, namely by passing to $\mathscr{E}/A^{n-1}$).

Using (1.6), we can now prove the following fundamental Jacobi-description of differentials. We let $U\rightarrowtail A^n$ be a coordinate neighbourhood and $f\colon U\to M$ a map into a Euclidean object. For convenience, scalars are multiplied on the right of M.

Proposition 1.4. *The differential of f*

$$df: U\times A^n\to M$$

has description

$$(a,b)\mapsto\sum_{j=1}^{n}\frac{\partial f}{\partial x_j}(a)\cdot b_j$$

where $b=(b_1,\ldots,b_n)$.

Proof. Again using (1.2), it suffices to prove commutativity of the diagram

$$\begin{array}{ccc} U\times A^n & \xrightarrow{\ \tilde f\ } & M\times M \\ \downarrow{\scriptstyle \alpha_U} & & \downarrow{\scriptstyle \alpha} \\ U^D & \xrightarrow[f^D]{} & M^D \end{array}$$

where $\tilde f$ has description

$$\langle a,b\rangle\mapsto\langle f(a),\sum\frac{\partial f}{\partial x_j}(a)\cdot b_j\rangle.$$

This is done by passing to exponential adjoints, so we should prove that the following diagram is commutative

$$\begin{array}{ccc} U\times A^n\times D & \xrightarrow{\ \tilde f\times D\ } & M\times M\times D \\ \downarrow{\scriptstyle \check\alpha} & & \downarrow{\scriptstyle \check\alpha_M} \\ U & \xrightarrow[f]{} & M \end{array}$$

which in elementwise terms says

$$f(a+d\cdot b)=f(a)+d\cdot\left(\sum\frac{\partial f}{\partial x_j}(a)\cdot b_j\right). \tag{1.7}$$

To prove this equation, we write $b = \sum b_j e_j$ and use (1.6) repeatedly on the left hand side of (1.7), not only for f but also for the $\partial f/\partial x_j$. Whenever a term contains two factors d, it vanishes, since $d^2 = 0$, whence in the final result no iterated partial derivatives

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$$

occur, and we end up with the right hand side of (1.7). This proves the proposition.

2. One-variable Taylor Series

To have some Taylor Theorems work in the present setting, we need to strengthen slightly the assumptions on the surrounding category $\mathscr{E}$, and to make the (strong) assumption that the ring object A considered is in fact an algebra over the rationals; so for any natural number $p \neq 0$, we have a map

$$\frac{1}{p} : A \to A,$$

with the expected properties.

To state the assumptions of categorical nature, we let for each natural number k $D_k \rightarrowtail A$ denote the equalizer of the two maps $A \to A$ given by the descriptions $a \mapsto a^{k+1}$ and $a \mapsto 0$, respectively. We shall assume that for each n the object A^{Dk} exists and that the map

$$\alpha_k : \underbrace{A \times \cdots \times A}_{k+1 \text{ copies}} - A^{D_k}$$

whose exponential adjoint $\check{\alpha}_k$ has description

$$\langle\langle a_0, \ldots, a_k \rangle, d\rangle \mapsto \sum_{j=0}^{k} a_j \cdot d^j,$$

is invertible. So we say that A is a ring object of line type in the extended sense.

Also, if M is an A-module object such that M^{D_k} exists for each $k = 1, 2, \ldots,$ we say that M is Euclidean-in-the-extended sense if the maps

$$\alpha_{k,M} : \underbrace{M \times \cdots \times M}_{k+1 \text{copies}} \to M^{D_k}$$

whose exponential adjoint have description

$$\langle\langle m_0, \cdots, m_k \rangle, d\rangle \mapsto \sum_{j=0}^{k} d^j \cdot m_j$$

are invertible.

Finally, $u: U \rightarrowtail M$ is called subeuclidean in the extended sense if $\alpha_{k,M}$ induces a map

$$\alpha_{k,U}: U \times \underbrace{M \times \cdots \times M}_{k \text{ copies}} \to U^{D_k}.$$

Clearly $D_k \subseteq D_{k+1}$; the restriction map $M^{D_{k+1}} \to M^{D_k}$ then makes the square

$$\begin{array}{ccc} M^{k+2} & \xrightarrow{\text{proj}_{0,\ldots,k}} & M^{k+1} \\ \alpha_{k+1}\downarrow & & \downarrow\alpha_k \\ M^{D_{k+1}} & \longrightarrow & M^{D_k} \end{array} \tag{2.1}$$

commutative. This generalizes (1.2) from $k=0$ to arbitrary k, since $D_1 = D$ and $D_0 = (\mathbb{1} \xrightarrow{\ulcorner 0 \urcorner} A)$.

We evidently have maps

$$D_k \times D_k \xrightarrow{\cdot} D_k \tag{2.2}$$

and

$$D_p \times D_q \xrightarrow{+} D_{p+q}, \tag{2.3}$$

restrictions of multiplication $A \times A \to A$ and addition $A \times A \to A$, respectively. (To see (2.3), note that if $d^{p+1}=0$ and $e^{q+1}=0$ then

$$(d+e)^{p+q+1} = \sum \binom{r+s}{r} d^r e^s$$

summed over r, s with $r+s=p+q+1$. But then each term in the sum contains either d in a power $\geq p+1$ or e in a power $\geq q+1$ and thus vanishes). Similarly, or by iteration of (2.3), we have maps

$$\underbrace{D_1 \times \cdots \times D_1}_{k} \to D_k \tag{2.4}$$

given by the description

$$\langle d_1, \cdots, d_k \rangle \mapsto \sum d_j.$$

Out of $\alpha = \alpha_1$, we can manufacture a map σ which is an isomorphism since α_1 is:

$$A^{2k} \xrightarrow{\sigma} A^{D_1 \times \cdots \times D_1} \tag{2.5}$$

(k copies of D_1 in the exponent), given as the exponential adjoint of the map $\check{\sigma}$ with description

$$\langle \{a_H\}_{H \subseteq [k]}, d_1, \ldots, d_k \rangle \overset{\check{\sigma}}{\mapsto} \sum_H a_H \cdot \prod_{h \in H} d_h.$$

(Note that we identify elements of 2^k with subsets of $[k] = \{1, 2, \ldots, k\}$). For $k=1$, the σ is just α_1.

Let M be a Euclidean object (in the extended sense). We shall describe in elementwise terms the unique map $\tilde{+}$ which will make the following diagram commutative:

$$\begin{array}{ccc} M^{k+1} & \xrightarrow{\tilde{+}} & M^{2^k} \\ {\scriptstyle \alpha_k}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \sigma} \\ M^{D_k} & \xrightarrow[M^+]{} & M^{D_1\times\cdots\times D_1} \end{array} \tag{2.6}$$

Proposition 2.1. *The map* $\tilde{+}: M^{k+1} \to M^{2^k}$ *given by the description*

$$\langle m_0, \ldots, m_k\rangle \mapsto \{H \mapsto j! \cdot m_j\}$$

(*where* j *is the number* $|H|$ *of elements in* $H \subseteq [k]$) *makes* (2.6) *commutative.*

(Again, an element in M^{2^k} is described by assigning to each element H of 2^k i.e. to each subset H of $[k]$, an element of M.)

Proof. We pass to exponential adjoints with (2.6). The counterclockwise composite then yields (denoting $D_1 \times\cdots\times D_1$ by $\mathscr{D}$)

$$\begin{aligned} & ev_{\mathscr{D}} \circ (M^+ \times \mathscr{D}) \circ (\alpha_k \times \mathscr{D}) \\ &\quad = ev_{D_k} \circ (M^{D_k} \times +) \circ (\alpha_k \times \mathscr{D}) \\ &\quad = ev_{D_k} \circ (\alpha_k \times D_k) \circ (M^{k+1} \times +) \\ &\quad = \check{\alpha}_k \circ (M^{k+1} \times +). \end{aligned}$$

The clockwise composite yields

$$ev_{\mathscr{D}} \circ (\sigma \times \mathscr{D}) \circ (\tilde{+} \times \mathscr{D}) = \check{\sigma} \circ (\tilde{+} \times \mathscr{D}).$$

So it suffices to prove commutative the diagram

$$\begin{array}{ccc} M^{k+1} \times D_1 \times\cdots\times D_1 & \xrightarrow{\tilde{+}\times\mathscr{D}} & M^{2^k} \times D_1 \times\cdots\times D_1 \\ {\scriptstyle M^{k+1}\times +}\downarrow & & \downarrow{\scriptstyle \check{\sigma}} \\ M^{k+1} \times D_k & \xrightarrow[\check{\alpha}_k]{\qquad\qquad} & M, \end{array} \tag{2.7}$$

which we do elementwise: Consider an element

$$\langle m_0, \ldots, m_k, d_1, \ldots, d_k\rangle \tag{2.8}$$

of the domain (so $d_1^2 = \ldots = d_k^2 = 0$). Taking it counterclockwise yields

$$\sum_{j=0}^{k} m_j \cdot (d_1 + \cdots + d_k)^j = \sum_{j=0}^{k} m_j \cdot \Big(\sum_{\substack{H \subseteq [k] \\ |H| = j}} j! \prod_{h \in H} d_h \Big), \tag{2.9}$$

since, when multiplying out $(d_1+\ldots+d_k)^j$ all terms with a repeated factor vanish, whence a term can be identified with a subset H of $[k]$ with j elements. Going clockwise with the element (2.8) yields, by definition of $\tilde{+}$

$$\check{\sigma}(\langle\{H\mapsto j!\cdot m_j\}, d_1,\ldots,d_k\rangle)=\sum_H j!m_j\prod_{h\in H} d_h$$

(j denoting the number of elements in $H\subseteq[k]$), which is the same as (2.9), and the proposition is proved.

Similarly, one may prove that the unique map $\tilde{+}$ which makes the diagram

$$\begin{array}{ccc} M^{n+m+1} & \xrightarrow{\tilde{+}} & (M^{n+1})^{m+1} \\ {\scriptstyle \alpha_{n+m}}\downarrow\cong & & \downarrow\cong \\ M^{D_{n+m}} & \xrightarrow[A^{+}]{} & M^{D_n\times D_m} \end{array} \tag{2.10}$$

commutative is given by

$$\langle m_0,\ldots,m_{n+m}\rangle \overset{\tilde{+}}{\mapsto} \left\{\binom{p+q}{p} m_{p+q}\right\}_{\substack{p=0,\ldots,n\\ q=0,\ldots,m}}$$

(the right hand vertical map in (2.10) being again an isomorphism derived from α_n and α_m).

Let $U\rightarrowtail A$ be subeuclidean. Then, as observed in Section 1, U is stable under addition of elements of D (in fact, since U is assumed to be subeuclidean in the extended sense, U is stable under addition of elements of any D_k). Thus, by iteration, we also have addition maps

$$U\times\underbrace{D_1\times\cdots\times D_1}_{n}\xrightarrow{+} U.$$

Let $f:U\to M$ with M Euclidean. We can then use first Taylor Lemma (1.6), to conclude equality of the two maps

$$U\times\underbrace{D_1\times\cdots\times D_1}_{k}\to M,$$

whose descriptions are the two sides in

$$f(a+d_1+\cdots+d_k)=\sum_{H\subseteq[k]} f^{(j)}(a)\cdot\prod_{h\in H} d_h,\quad (j=|H|), \tag{2.11}$$

(where $f^{(j)}=(f^{(j-1)})'$, and where prime denotes derivation along $\hat{+}:U\to U^D$ or, equivalently, $g'=\partial g/\partial x_1$). To prove (2.11) formally, we proceed by induction:

$$\begin{aligned} f(a+d_1+\ldots+d_k)&=f((a+d_1+\cdots+d_{k-1})+d_k)\\ &=f(a+d_1+\cdots+d_{k-1})+f'(a_1+\cdots+d_{k-1})\cdot d_k, \end{aligned}$$

(using (1.6)), and by induction hypothesis on each of the two terms, we get (j denoting number of elements in H)

$$= \sum_{H\subseteq[k-1]} f^{(j)}(a)\cdot \prod_{h\in H} d_h + \sum_{H\subseteq[k-1]} f^{(j+1)}(a)\cdot \prod_{h\in H} d_h\cdot d_k. \tag{2.12}$$

This equals the right hand side of (2.11); for a set $H\subseteq[k]$ *either* does not contain k, and the term corresponding to such H occurs (exactly once) in the first sum in (2.12); *or* it does contain k, in which case the term corresponding to H occurs (with index $H'=H\setminus\{k\}$) in the second sum in (2.12) (exactly once).

Recall the assumptions of this paragraph: A is a ring object of line type in the extended sense, i.e. $A^{D_k}\cong A^{k+1}$ via α_k, and A is an algebra over the rationals. Under these assumptions, we have

Theorem 2.2. (Taylor). *Let $U\rightarrowtail A$ be subeuclidean, and let $f:U\to M$ be any map into a Euclidean object. Then the two maps*

$$U\times D_k \rightrightarrows M$$

whose descriptions are the two sides in (2.13), *agree*:

$$f(a+d)=\sum_{j=0}^{k}\frac{1}{j!}d^j\cdot f^{(j)}(a). \tag{2.13}$$

Proof. Take the exponential adjoint of the two maps in question; this is a diagram

$$U\rightrightarrows M^{D_k}$$

to prove it commutative, it suffices to prove

$$U\rightrightarrows M^{D_k}\xrightarrow{M^+} M^{D_1\times\cdots\times D_1} \tag{2.14}$$

commutative (k copies of D_1), because it is evident from the algebra-over-the-rationals assumption and Proposition 2.1 that

$$M^{D_k}\xrightarrow{M^+} M^{D_1\times\cdots\times D_1}$$

is monic. To prove (2.14) commutative, we go to exponential adjoints once more, and obtain the diagram

$$U\times D_1\times\cdots\times D_1\xrightarrow[U\times +]{} U\times D_k\rightrightarrows M \tag{2.15}$$

obtained from $U\times D_k\rightrightarrows M$ by multiplying $U\times +$ on the left. Now we prove (2.15) commutative by means of elements. We must prove the equality (for $d_1^2=\cdots=d_k^2=0$):

$$f(a+d_1+\cdots+d_k)=\sum_{j=0}^{k}\frac{1}{j!}(d_1+\cdots+d_k)^j f^{(j)}(a). \tag{2.16}$$

Essentially, we have done the work already in formula (2.11). We just have to see that the right hand sides of (2.11) and (2.16) agree. In the j'th term of the right hand side of (2.16) the term

$$\frac{1}{j!} d_{h_1} \cdot \ldots \cdot d_{h_j} \cdot f^{(j)}(a)$$

(for $h_1 < h_2 < \cdots < h_j$) occurs $j!$ times and the sum of these $j!$ terms correspond exactly to the H summand $f^{(j)}(a) \cdot \prod_{h \in H} d_h$ where $H = \{h_1, \ldots, h_j\}$. Terms with a repeated factor d_i vanish because $d_i^2 = 0$ by assumption. The theorem is proved.

3. Several variables—global sections case

A *global section* of an object M in a category $\mathscr{E}$ is a map $b: \mathbb{1} \to M$. For any X in $\mathscr{E}$, b will also denote the composite $X \to \mathbb{1} \xrightarrow{b} M$. Arbitrary maps $X \to M$ may be reinterpreted as global sections in $\mathscr{E}/X$. Now our axioms are stable under passage from $\mathscr{E}$ to $\mathscr{E}/X$ in a sense that will be explained in Section 4. Therefore the results of the present section, which deal with global sections, are not so special as it may seem.

With the assumptions of Section 2 on the category $\mathscr{E}$ and the ring object of line type A, we consider a Euclidean object N and a subeuclidean $U \rightarrowtail N$. Then any global section b: $1 \to N$ gives rise to a tangent vector field $\mathbf{b}.$ on U:

$$\mathbf{b}.: U \to U^D$$

is given as exponential adjoint of the map $U \times D \to U$ with description

$$\langle a, d \rangle \mapsto a + d \cdot b;$$

$\mathbf{b}.$ is a "constant vector field". With such N, b and U we have

Proposition 3.1. *Let M be Euclidean and f: $U \to M$ a map. Then*

$$df_a(b) = D_{\mathbf{b}}.f(a) \quad \forall a \in U.$$

Proof. The map with description $a \mapsto df_a(b)$ is by definition of differentials the composite

$$U \xrightarrow[\cong]{} U \times \mathbb{1} \xrightarrow[U \times b]{} U \times N \xrightarrow[\alpha_U]{} U^D \xrightarrow[f^D]{} M^D \xrightarrow[\gamma]{} M.$$

However, it is easily seen that the composite of the three first maps here is just the vector field $\mathbf{b}.$, whence the proposition. Note that if $b = 1 \in A$, then $D_{\mathbf{b}}.f = f'$.

Proposition 3.2. *If f: $N \to M$ is an A-linear map between Euclidean objects, then df: $N \times N \to M$ is just $f \circ \mathrm{proj}_1$.*

Proof. By the characterizing property (1.5) of df, it suffices to prove

$$\begin{array}{ccc} N\times N & \xrightarrow{f\times f} & M\times M \\ {\scriptstyle\alpha_N}\downarrow & & \downarrow{\scriptstyle\alpha_M} \\ N^D & \xrightarrow[f^D]{} & M^D \end{array}$$

commutative. But when taking the exponential adjoint diagram, commutativity is immediate from the A-linearity of f.

Now let N be a Euclidean object and $\boldsymbol{b}$ a global section of N. We consider the linear map $h: A \to N$ given by the description $t \mapsto t \cdot \boldsymbol{b}$. Then we have for any $s \in A$

$$\boldsymbol{b} = h(1) = dh_s(1) = D_1.h(s) = h'(s)$$

by Proposition 3.2 and 3.1, so h' is constant $\boldsymbol{b}$. More generally, let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two global sections of N, and consider the map $h: A \to N$ with description

$$h: t \mapsto \boldsymbol{a} + t \cdot \boldsymbol{b}.$$

Since (Proposition 1.2) the derivative of a constant is 0, and derivation is additive, we conclude that h' is constant $\boldsymbol{b}$, or alternatively (using Proposition 3.1 again) that

$$dh_s(t) = t \cdot dh_s(1) = t \cdot D_1.h(s) = t \cdot \boldsymbol{b} \quad \forall s. \tag{3.1}$$

Assume now further that $U \rightarrowtail N = A^n$ is subeuclidean and that $\boldsymbol{a}: 1 \to N$ factors through U. Let $f: U \to M$ be a map into an Euclidean M. We shall further assume that there exists a subeuclidean $W \rightarrowtail A$ containing 0 so that h maps W into U (the $D_\infty = \bigcup D_k$ considered later will serve for W). Let g denote the composite

$$W \xrightarrow{h} U \xrightarrow{f} M.$$

Applying the one-variable Taylor Theorem 2.2, we obtain, for $d \in D_k$,

$$f(\boldsymbol{a} + d \cdot \boldsymbol{b}) = g(d) = \sum_{j=0}^{k} \frac{d^j}{j!} g^{(j)}(0).$$

We now compute the $g^{(j)}(0)$ in terms of f. We have $g^{(0)}(t) = g(t) = f(\boldsymbol{a} + t \cdot \boldsymbol{b})$. We prove by induction

$$\forall t: g^{(j)}(t) = \sum_{i_1=1}^{n} \cdots \sum_{i_j=1}^{n} D_{i_j} \cdots D_{i_1} f(\boldsymbol{a} + t\boldsymbol{b}) \cdot b_{i_1} \cdots b_{i_j}$$

(where b_r is r'th coordinate of $\boldsymbol{b}$). Assume that this formula holds; we then prove the similar formula for $g^{(j+1)}(t)$. Since differentiation is a derivation, and the derivative of a constant (like b_r) is zero, we can differentiate $g^{(i)}$ termwise, whence (by changing notation, denoting $D_{i_j} \cdot \ldots \cdot D_{i_1} f$ by f), we only have to prove

$$g'(t) = \sum_{i=1}^{n} D_i f(\boldsymbol{a} + t \cdot \boldsymbol{b}) \cdot b_i \tag{3.2}$$

(with g, f, $\boldsymbol{a}$ and $\boldsymbol{b}$ as above). Now, using Proposition 3.1,

$$\begin{aligned} g'(t) &= D_1 . g(t) = dg_t(1) \\ &= d(f \circ h)_t(1) \\ &= df_{h(t)}(dh_t(1)) \qquad \text{(by chain rule)} \\ &= df_{\boldsymbol{a}+t\cdot\boldsymbol{b}}(\boldsymbol{b}) \qquad \text{(by (3.1))} \end{aligned}$$

which equals the right hand side of (3.2) by Proposition 1.4. We have thus proved:

Theorem 3.3. *Let $U \rightarrowtail A^n$ be subeuclidean, $f: U \to M$ a map into a Euclidean object, and $\boldsymbol{a}$ and $\boldsymbol{b}$ global sections of U and A^n, respectively. Then, for $d \in D_k$ we have*

$$f(\boldsymbol{a}+d\cdot\boldsymbol{b}) = \sum_{j=0}^{k} \frac{d^j}{j!} \sum_{i_1=1}^{n} \cdots \sum_{i_j=1}^{n} D_{i_1} \cdots D_{i_j} f(\boldsymbol{a}) \cdot b_{i_1} \cdot \ldots \cdot b_{i_j}. \tag{3.3}$$

This expression can be somewhat simplified because we can prove interchangeability of two partial differentiation processes.

In view of a later use, we shall here remark that if $f: U \to M$ is a map from a subeuclidean $U \subseteq A^k$ into a Euclidean M, and $\boldsymbol{a}: \mathbf{1} \to U$ is a global section, then we can define $D_i f(\boldsymbol{a})$, which is a global section $\mathbf{1} \to M$; but this global section could have been defined even if f had only been defined on the subobject $\boldsymbol{a} + D_1 \cdot \boldsymbol{e}_i$ of U. For, as above, $D_i f(\boldsymbol{a})$ is the derivative at 0 of a certain map $g: W \to M$, and (for any $g: W \to M$) $g'(0)$ can be defined just knowing g on D_1, namely as

$$\mathbb{1} \longrightarrow D_1^{D_1} \xrightarrow{g^{D_1}} M^{D_1} \xrightarrow{\gamma} M$$

where the first map is the exponential adjoint of the identity map. More generally, if $g: D_r \to M$, the p'th derivative of g can be defined as a map $D_{r-p} \to M$.

Thus, all the iterated partial derivatives in (3.3) only depend on having defined on

$$\boldsymbol{a} + (\underbrace{D_k \times \cdots \times D_k}_{n}).$$

4. Several variable Taylor Theorems

On basis of (3.3), it is a matter of pure category theory to prove the following Taylor Theorem (a different one appears as Theorem 4.3):

Theorem 4.1. *Let $U \rightarrowtail A^n$ be subeuclidean, and $f: U \to M$ a map into a Euclidean object. Then the two maps*

$$U \times A^n \times D_k \to M$$

whose descriptions are the two sides of (3.3) *agree.*

Proof. Let $\boldsymbol{a}: X \to U$, $\boldsymbol{b}: X \to A^n$, $\boldsymbol{d}: X \to D_k$ be elements with common domain. It suffices to prove (3.3) for these elements. Let us denote the functor $\mathscr{E} \to \mathscr{E}/X$ given by

$$C \mapsto \begin{array}{c} C \times X \\ \big\downarrow {\scriptstyle \text{proj}} \\ X \end{array}$$

by $(\ )_X$. It commutes with inverse limits and preserves those exponentials that exist. Therefore if A is a ring object of line type in $\mathscr{E}$, then A_X is a ring object of line type in $\mathscr{E}/X$. Also, $(\ )_X$ preserves differentiation and derivation. Now $\boldsymbol{a}: X \to U$ gives rise to

$$\begin{array}{ccc} X & \xrightarrow{\langle \boldsymbol{a}, \text{id}\rangle} & U \times X \\ & {\scriptstyle \text{id}}\searrow \quad \swarrow{\scriptstyle \text{proj}} & \\ & X & \end{array}$$

that is, to a global section $\tilde{\boldsymbol{a}}: \mathbb{1}_X \to U_X$. Similarly for $\boldsymbol{b}$ and $\boldsymbol{d}$. By (3.3) applied in $\mathscr{E}/X$ we conclude equality of the two global sections of M_X (in $\mathscr{E}/X$) which are denoted by the two sides of (3.3) with a $\tilde{}$ placed on each of the letters $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{d}$ in there. So we have equality of two maps $\mathbb{1}_X \to M_X$. Applying the obvious forgetful functor $\mathscr{E}/X \to \mathscr{E}$ we get equality of two maps $X \to M \times X$ in $\mathscr{E}$, and taking projection onto first factor, we conclude the equality of two maps $X \to M$, as desired. But are the two maps the desired maps? Yes, because $(\ )_X$ preserves differentiation and the ring operations, meaning for instance equality of

$$\mathbb{1}_X \xrightarrow{\ \boldsymbol{a}+\boldsymbol{b}\ } U_X \xrightarrow{\ D_i(f_X)\ } M_X$$

and

$$\begin{array}{ccccc} X & \xrightarrow{\langle \boldsymbol{a}+\boldsymbol{b}, \text{id}\rangle} & U \times X & \xrightarrow{D_i f \times \text{id}} & M \times X \\ & {\scriptstyle \text{id}}\searrow & \big\downarrow {\scriptstyle \text{proj}} & \swarrow {\scriptstyle \text{proj}} & \\ & & X & & \end{array}$$

but applying the forgetful functor $\mathscr{E}/X \to \mathscr{E}$ to the latter, and then composing with proj: $M \times X \to M$ precisely yields $D_i f(\boldsymbol{a}+\boldsymbol{b})$.

Corollary 4.2. *Let i be a fixed index between* 1 *and n. Then the two maps $U \times D_k \to M$ with descriptions*

$$f(\boldsymbol{a} + d \cdot \boldsymbol{e}_i) = \sum_{j=0}^{k} \frac{d^j}{j!} D_i^{(j)} f(\boldsymbol{a}), \quad (d \in D_k)$$

agree.

Proof. Put $\boldsymbol{b} = \boldsymbol{e}_i$ in (3.3).

The technique employed in the proof of Theorem 4.1 also allows us to view partial differentiation as a special case of ordinary 1-variable differentiation. We confine ourselves to sketch this in a special case. Suppose given a map in $\mathscr{E}$

$$A \times N \xrightarrow{f} M$$

with M a Euclidean object. The partial derivative of f along the "first coordinate" A is by definition the derivative of f in the direction of the tangent vector field X_1: $A \times N \to (A \times N)^D$ given as exponential adjoint of the map

$$A \times N \times D \to A \times N$$

with description

$$\langle a, n, d \rangle \mapsto \langle a + d, n \rangle$$

(This is consistent with our earlier description of partial derivation, $\partial f/\partial x_1$ or $D_1 f$). However, out of f we can manufacture a map $\bar{f}$ in $\mathscr{E}/N$, namely

$$\begin{array}{ccc} A_N = A \times N & \xrightarrow{\langle f, \text{proj}_N \rangle} & M \times N = M_N \\ & \searrow^{\text{proj}} \quad {}^{\text{proj}}\swarrow & \\ & N & \end{array}$$

We thus have a map $\bar{f}$: $A_N \to M_N$ in $\mathscr{E}/N$ with domain a ring object of line type and with Euclidean codomain. It can be differentiated to yield

$$\bar{f}': A_N \to M_N,$$

and reinterpreting this map as a map $A \times N \to M \times N$ and composing with proj_M: $M \times N \to M$ yields $D_{X_1} f$.

There are some "neighbourhoods" of $\mathbf{0} \in A^m$ which we want to consider: let $D(m)_k$ denote the extension

$$\{(d_1, \ldots, d_m) | \text{ any product of } k+1 \text{ copies of the } d_i\text{'s is } 0\}.$$

(Formally, this is an intersection of m^{k+1} subobjects of A^m. In the case $m = 2$, $k = 1$, it is the intersection of $D_1 \times A$, $A \times D_1$, and of the following subobject of $A \times A$ (counted twice):

$$\{(d_1, d_2) \mid d_1 \cdot d_2 = 0\},$$

(which just is notation for the equalizer of the multiplication map $A \times A \to A$ and the constant-0 map $A \times A \to A$).

Clearly

$$D(m)_k \subseteq D_k^m \tag{4.1}$$

but also

$$D_k^m \subseteq D(m)_h \tag{4.2}$$

for h sufficiently large ($h = m \cdot k$ will do).

We can now prove a Taylor Theorem related to Theorems 4.1 and 3.3.

Theorem 4.3. *Let $f: U \to M$ where U is a subeuclidean object of A^n, and M is any Euclidean object. Then the two maps*

$$U \times D(n)_k \to M$$

with descriptions

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle \mapsto f(\boldsymbol{a} + \boldsymbol{b})$$

and

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle \mapsto \sum_{j=0}^{k} \frac{1}{j!} \sum_{i_1=1}^{n} \cdots \sum_{i_j=1}^{n} D_{i_1} \cdots D_{i_j} f(\boldsymbol{a}) \cdot b_{i_1} \cdot \ldots \cdot b_{i_j}$$

agree.

Proof. This is straightforward using induction in n and Corollary 4.2. More precisely, let the induction hypothesis be that the theorem holds for $\boldsymbol{b}$ of form

$$(b_1, \ldots, b_r, 0, \ldots, 0).$$

We omit details.

5. Power series

In this section we add a weak assumption of purely categorical nature. It says that the increasing sequence of subobjects of A

$$D_1 \subseteq D_2 \subseteq \ldots \subseteq A$$

should have a colimit D_∞ which again is a subobject of A; this (filtering) colimit should further commute with finite inverse limits. This condition is satisfied in the ring classifier topos, because it is a topos, and in the category of formal schemes, because it is a locally finitely presented category. Also $D_\infty^{D_k}$ should exist for any k (this is again so in the two specifically mentioned categories; for they are cartesian closed).

We are then in a position to prove that D_∞ is a subeuclidean object (in the extended sense;—we assume throughout that A is a ring object of line type in the extended sense of Section 2). For each k, we have to prove existence of a factorization

$$\begin{array}{ccc} D_\infty \times A^k & \dashrightarrow & D_\infty^{D_k} \\ \downarrow & & \downarrow \\ A \times A^k & \xrightarrow[\alpha]{\cong} & A^{D_k} \end{array}$$

By passing to exponential adjoints, this is equivalent to proving that

$$\begin{aligned} &D_\infty \times A^k \times D_k \to A \\ &\langle t, a_1, \ldots, a_k, d\rangle \mapsto t + \sum_{j=1}^{k} a_j d^j \end{aligned} \tag{5.1}$$

factors through D_∞. Now

$$D_\infty \times A^k \times D_k = (\varinjlim D_j) \times A^k \times D_k = \varinjlim_j (D_j \times A^k \times D_k)$$

(by the limit-colimit commutation assumption), so that it suffices to prove that the right hand side of (5.1) is in D_∞ provided t is in D_j. To prove this is again an exercise with binomial coefficients. So

Proposition 5.1. *The subobject $D_\infty \rightarrowtail A$ is subeuclidean (in the extended sense).*

Borrowing a term, and some of the spirit, from non standard analysis, one would call D_∞ the monad around 0 in A ("the set of infinitesimally small elements"). It is in fact the smallest subeuclidean object containing 0. Do we have a similar "monad" in A^m for $m \geq 2$?

Proposition 5.2. *The product of m copies of D_∞ is subeuclidean in A^m, and is the smallest subeuclidean object of A^m containing* **0**. *It equals $\varinjlim_k D(m)_k$.*

Proof. For simplicity of notation, we do the case $m = 2$. By limit-colimit commutation

$$D_\infty \times D_\infty = \varinjlim_k (D_k \times D_k)$$

and like in the proof of Proposition 5.1, it suffices to see that

$$t + \sum_{j=1}^{n} b_j \cdot d^j \tag{5.2}$$

is in $D_\infty \times D_\infty$ when $t \in D_k \times D_k$, $d \in D_n$, and each $b_j \in A^2$. The i'th coordinate ($i = 1, 2$) of (5.2) is

$$t_i + b_{ji} \cdot d^j$$

which again is in some D_l. The minimality property is left to the reader (it will play no role in the sequel). That it equals $\lim_{\to} D(m)_k$ is immediate from (4.1) and (4.2).

If the "local behaviour near 0" of a function means its behaviour on the monad around 0, then the following theorem expresses that the Taylor Series expansion of a function precisely contains the information of its local behaviour.

Let $\mathscr{A}$ denote the ring of global sections $\mathbb{1} \to A$. It is an algebra over the rationals by the assumptions made.

Theorem 5.3. *There is a* 1—1 *correspondence between*

$$\hom_{\mathscr{E}}(\underbrace{D_\infty \times \cdots \times D_\infty}_{m}, A) \quad \text{and} \quad \mathscr{A}[\![X_1, \ldots, X_m]\!],$$

the ring of formal power series over $\mathscr{A}$ in m commuting variables. The correspondence is given by Taylor Series expansion, and is compatible with the ring-structure and substitution (to the extent the latter is defined—see Proposition 5.5).

Proof. Given $f: D^m \to A$; we associate to it the "Taylor Series"

$$\sum_{j=0}^{\infty} \frac{1}{j!} \sum_{i_1=1}^{m} \cdots \sum_{i_j=1}^{m} D_{i_1} \ldots D_{i_j} f(\mathbf{0}) \cdot X_{i_1} \cdot \ldots \cdot X_{i_j} \tag{5.3}$$

in the m commuting variables $X_1, \ldots, X_m$. Conversely, given a formal power series in $\mathscr{A}[\![X_1, \ldots, X_m]\!]$, we may write it uniquely in the form

$$\sum_{j=0}^{\infty} \sum_{i_1=1}^{m} \cdots \sum_{i_j=1}^{m} a_{i_1 \ldots i_j} \cdot X_{i_1} \cdot \ldots \cdot X_{i_j}$$

with the $a_{i_1 \ldots i_j}$ not depending on the ordering of the indices $i_1, \ldots, i_j$. The n'th partial sum $\sum_{j=0}^{n}$ in here is a polynomial of degree n with coefficients from $\mathscr{A} = \hom(\mathbb{1}, A)$, so defines a map

$$f_n: \underbrace{A \times \cdots \times A}_{m \text{ copies}} \to A.$$

It is clear that

$$f_n | D(m)_k = f_{n+1} | D(m)_k$$

for $n \geq k$. From this follows that the f_n's together describe a map

$$D_\infty^m = \varinjlim_k (D(m)_k) \xrightarrow{f} A.$$

The two processes described are inverse of each other. Let us start with a map $f: D_\infty^m \to A$. We form the series (5.3), and out of that, in turn, we construct a map $\bar{f}: D_\infty^m \to A$, as above. Since $D_\infty^m = \lim_\to(D(m)_k)$, to see $f = \bar{f}$, it suffices to see that f and $\bar{f}$ agree on $D(m)_k$. But this follows from Taylors Theorem 4.3.

Conversely, given a formal series of the form (5.4). We construct a function f: $D_\infty^m \to A$ as above and take its Taylor Series. As we have observed in the end of Section 3, the j'th partial derivatives of f in $\mathbf{0}$ only depend on the restriction of f to D_k^m, and this restriction is given by a polynomial of degree h (for suitable large h, say $\geq m \cdot k$), namely the h'th partial sum of the given series. Because partial differentiation is a derivation, it follows that a polynomial function equals its Taylor series (which terminates). Thus the series for f equals the given series.

(As a corollary of the proof here, we infer that

$$D_{i_1}D_{i_2}f = D_{i_2}D_{i_1}f;$$

this fact can also be established directly for any map $f: U \to M$ with M Euclidean).

It is evident that the process T ("taking Taylor series")

$$\hom_{\mathscr{E}}(D_\infty \times \cdots \times D_\infty, A) \to \mathscr{A}[\![X_1, \ldots, X_m]\!]$$

commutes with addition and multiplication by constants $\mathbf{1} \to A$. To investigate its behaviour with respect to multiplication, we consider T^{-1} instead; so let f and g be two formal series in $X_1, \ldots, X_n$ and $f \cdot g$ their (Cauchy-) product. We shall prove

$$T^{-1}(f \cdot g) = T^{-1}(f) \cdot T^{-1}(g).$$

The two sides here are maps $D_\infty \times \ldots \times D_\infty \to A$. To prove their equality, it suffices, by

$$\underbrace{D_\infty \times \cdots \times D_\infty}_{m} = \varinjlim_k D(m)_k$$

to prove their restrictions to $D(m)_k$ equal. But these restrictions are represented by polynomials of degree k (the k'th partial sum of the series in question), and the product of two maps described by polynomials equals the map described by the formal product of the polynomials (this is true in any category).

This proves the theorem, except for the statement concerning substitution. To make this statement precise, let

$$\hom^+(D_\infty \times \cdots \times D_\infty, A) \subseteq \hom(D_\infty \times \cdots \times D_\infty, A)$$

denote the subset consisting of those maps f for which $f(\mathbf{0}) = 0$. Then T of such an f lies in $\mathscr{A}_+[\![X_1, \ldots, X_n]\!]$, the set of formal series with zero constant term.

Lemma 5.4. *Let $f(0)=0$. Then f factors through $D_\infty \subseteq A$.*

Proof. Since the domain of f is a direct limit $\varinjlim_k D(m)_k$, it suffices to prove that the restriction of f to each $D(m)_k$ factors through $D_\infty \subseteq A$. But this restriction is given by a polynomial, namely the k'th partial sum of the Taylor Series for f, and has 0 constant term. So it is a sum of terms where each term contains at least one factor from D_k. Thus raised to a sufficiently high power, the sum vanishes.

We can now consider two algebraic theories, $\mathcal{T}^+$ and $\hat{\mathcal{T}}_0$ where

$$\mathcal{T}^+(m,1) = \hom^+(\underbrace{D_\infty \times \cdots \times D_\infty}_{m}, D_\infty)$$

$$= \hom^+(D_\infty \times \cdots \times D_\infty, A),$$

and where $\hat{\mathcal{T}}_0$ is is the algebraic theory of formal power series without constant term over $\mathcal{A}$ (as considered in [2], say). Then Taylor-Series formation defines maps

$$\mathcal{T}^+(m,1) \xrightarrow{T_m} \hat{\mathcal{T}}_0(m,1)$$

and the statement about compatibility of T with substitution can be given the following precise formulation.

Proposition 5.5. *The maps T establish an isomorphism between the algebraic theories $\mathcal{T}^+$ and $\hat{\mathcal{T}}_0$.*

Proof. Since each T_m is bijective (and $\mathcal{A}$-linear) it suffices to prove it for T^{-1}. Again, as for the multiplicative property of T, it suffices to prove it for polynomials. But it is well known that formal substitution of polynomials correspond to composition of the maps described by the polynomials on any ring object in any category with finite products—just because the algebraic theory consisting of polynomials and their formal substitution is the algebraic theory of commutative rings.

A wealth of corollaries about existence of maps defined on D_∞^m follow from Proposition 5.5 and well known theorems about formal power series. Thus we can have inverse- and implicit functions theorems for maps $D_\infty^n \to D_\infty^m$, Weierstrass preparation theorem, and existence and uniqueness of solutions for differential equations. We just give one of these as an example, the formal inverse function theorem.

Theorem 5.6. *Let $f\colon D_\infty^n \to D_\infty^n$ be a map with $f(\mathbf{0}) = \mathbf{0}$. Then there is an inverse map g if and only if*

$$df_{\mathbf{0}}\colon A^n \to A^n$$

is invertible.

Proof. f is given by n maps f_i: $D_\infty^n \to D_\infty$ with $f_i(\mathbf{0}) = 0$, thus by n series in n variables $X_1, \ldots, X_n$, and with 0 constant term. It is well known that such an n-tuple of series is invertible if and only if the linear terms form an invertible $n \times n$ matrix. But this matrix is precisely the matrix for $df_\mathbf{0}$, using Proposition 1.4.

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