Theory of characteristics for first order partial differential equations

Anders Kock

Dabei sehen wir von unendlich kleinen Grössen höhere Ordnung ab.
Lie [7] p. 523

Introduction

The present note makes no claim of originality; it is a "conspectus" of some of the classical theory of characteristics for 1st order PDEs, as expounded geometrically by Lie and elaborated by Klein, notably §72 in [1]. These authors use extensively a synthetic geometric language, but ultimately describe notions rigourously only by presenting them in analytic terms. Our approach describes the notions, like "united position" ("vereinigte Lage") and "characteristic", rigourously in pure synthetic and coordinate free terms, and introduces coordinates only at a later point, when it comes to proving some of the relations between the notions introduced.

So we are not claiming that describing the notions synthetically is an effective tool for *proving*; usually, coordinates are better suited for this. The virtue of the synthetic descriptions are, as also appears from the work of Monge, Lie, Klein, et. al., that it gives a geometric language to speak about geometric entities, and in particular, make them coordinate free from the outset.

The particular version of synthetic language that we use is that of Synthetic Differential Geometry (SDG), as in [2], [3] say, and notably as in [5], where the main synthetic relation is the first and second order neighbour relation of a scheme or of a manifold, . Such neighbourhoods (of various orders) was first considered in French algebraic geometry and global analysis in the 1950s, see by Malgrange et al. see e.g. [8]. We denote these relations (of order 1 and 2) by the symbol $\sim_1$ (or just $\sim$) and $\sim_2$, respectively. They are reflexive symmetric relations on a scheme resp. manifold, and these relations are preserved by any map in the relevant category (whose objects we call manifolds). The manifolds that we consider are derived from an ambient 3-dimensional manifold M (where "dimension" refers to the analytic model, ultimately: $M = R^3$). "Curves" and "surfaces" are manifolds of dimension 1 and 2, respectively.

The object of kth order neighbours of a point x in a manifold M is denoted $\mathfrak{M}_k(x)$, i.e. $\mathfrak{M}_k(x) = \{y \in M \mid y \sim_k x\}$. One has that $x \sim_1 y \sim_1 z$ implies $x \sim_2 z$. Equivalently,

$$y \in \mathfrak{M}_1(x) \text{ implies } \mathfrak{M}_1(y) \subseteq \mathfrak{M}_2(x). \tag{1}$$

The axiomatics used for these neighbourhoods is essentially the "Kock-Lawvere" (KL) axiom scheme, which we shall quote when needed. The basic manifold is the number line R; here $x \sim_k y$ iff $(y-x)^{k+1} = 0$. In R^n, the set $\mathfrak{M}_1(0)$ is also denoted $D(n)$, and $\mathfrak{M}_k(0)$ is denoted $D_k(n)$. See the Appendix for the algebraic description.

Since all maps preserve $\sim$, it follows that in $X \times Y$, we have $\mathfrak{M}_1(x,y) \subseteq \mathfrak{M}_1(x) \times \mathfrak{M}_1(y)$ (where $x \in X,\ y \in Y$). The converse inclusion does not hold in general. However, if $K \subseteq X \times Y$ is the graph of a map $f : X \to Y$, we have $(\mathfrak{M}_1(x) \times Y) \cap K \subseteq \mathfrak{M}_1(x,y)$. For, any map preserves $\sim$, and this applies to the map $x \mapsto (x, f(x))$ from X to K, as well as to the inclusion $K \subseteq X \times Y$.

1 Surface elements and calottes

Let M be a 3-dimensional manifold.

A *surface element at* x is a set $P \subset M$ of the form $\mathfrak{M}_1(x) \cap F$, where $F \subset M$ is a surface (2-dimensional submanifold of M) containing x. Similarly a *calotte at* x is a set $K \subset M$ of the form $\mathfrak{M}_2(x) \cap F$ where $F \subset M$ is a surface containing x. (The notion of calotte is from [1] p. 281.)

If K is a calotte at x, it is clear that $\mathfrak{M}_1(x) \cap K$ is a surface element P at x, called the *restriction* of K, and similarly, K is an *extension* calotte of P.

Surface elements by definition come equipped with base points, since we only defined the notion of surface element *at* a point; similarly for calottes.

More generally,

Proposition 1 *If K is a calotte at x and if $y \in \mathfrak{M}_1(x) \cap K$, then $\mathfrak{M}_1(y) \cap K$ is a surface element.*

Proof. Let F is any surface with $K = \mathfrak{M}_2(x) \cap F$. Since $\mathfrak{M}_1(y) \subseteq \mathfrak{M}_2(x)$ by (1),

$$\mathfrak{M}_1(y) \cap F \subseteq \mathfrak{M}_2(x) \cap F = K$$

so $\mathfrak{M}_1(y) \cap F = \mathfrak{M}_1(y) \cap K$. So F witnesses that $\mathfrak{M}_1(y) \cap K$ is a surface element.

So a calotte K at x defines a family of surface elements, namely the family of pointed sets $\mathfrak{M}_1(y) \cap K$ (pointed by y) for y ranging over $\mathfrak{M}_1(x)$.

The surface elements P' coming about from K in this way are said to *belong to* K; and the base point of a P' and the base point of K are (first order) neighbours.

One has the following "dimension principle":
for two surface elements P and P' with same base point, $P' \subseteq P$ implies $P = P'$.

This follows in SDG from the fact from linear algebra that for two linear subspaces V and V' of same dimension, $V' \subseteq V$ implies $V' = V$.

One could also use the terms "1-jet (resp. 2-jet) of a surface" for surface elements, respectively calottes, in M. We denote the manifold of surface elements, respectively the manifold of calottes, by the symbols $S_1(M)$, respectively $S_2(M)$. We have surjective submersions

$$S_2(M) \to S_1(M) \to M.$$

The dimensions of these manifolds are 8, 5, and 3, respectively, cf. Section 5. The manifold $S_1(M)$ may be described as the projectivization $P(T^*M)$ of the cotangent bundle $T^*(M) \to M$.

Note that the restriction of K belongs to K; a surface element which belongs to K is the restriction of K iff its base point is x.

Proposition 2 *Let $P \sim P'$ be contact elements in M with base points $x \sim y$, respectively. Let K be a calotte with restriction P. If P' belongs to K, then $P' = \mathfrak{M}_1(y) \cap K$.*

Proof. Clearly, $P' \subseteq \mathfrak{M}_1(y) \cap K$. By the dimension principle, it therefore suffices to know that $\mathfrak{M}_1(y) \cap K$ is a contact element, and this we know from Proposition 1.

2 The contact distribution $\approx$

We consider a general 3-dimensional manifold M, and the corresponding 5-dimensional manifold $S_1(M)$ of its surface elements.

Being a manifold, $S_1(M)$ carries a (1st order) neighbour relation $\sim$. It carries a further structure, namely a reflexive symmetric relation $\approx$ refining $\sim$, and called "united position[1]" ("vereinigte Lage", in the terminology og Lie and Klein): if P and Q are neigbour surface elements with base points p and q, respectively, we say that

$$P \approx Q \quad \text{if} \quad q \in P.$$

This is almost a literal translation of the definition in Lie [7] p. 523: "a surface element is in united position with another one if the [base] point of the latter lies

[1] In modern treatments, the structure "united position" is presented as subordinate to the canonical *contact manifold* structure which the cotangent bundle T^*M carries – a certain canonical 1-form. However, $P(T^*M)$ does not carry a canonical 1-form (only "modulo a scalar factor"), and our description (i.e. Lie's) of $\approx$ is purely geometric.

in the plane of the former". It is not immediate from our definition that $\approx$ is a *symmetric* relation, but this can be proved if we, in Lie's verbal rendering ([7] p. 523), *"ignore infinitesimally small quantities of higher order"*. In our context, the "ignored quantities" are not only ignored, but are *equal* to 0, using $P \sim Q$, as the coordinate calculation below (5) will reveal. (Such refinement $\approx$ of $\sim$ is a synthetic way to describe a geometric *distribution*, cf. [3] or [5] for an elaboration of this viewpoint.)

In Lie's treatment, the "united position" structure $\approx$ on the manifold $S_1(M)$ gives rise to another one, namely the notion of *strip*: this is a curve C in $S_1(M)$, where neighbour elements in C are in united position; for P and P' in C, $P \sim P'$ implies $P \approx P'$.

Let F be a surface in M. Since the passage from points x in F to the corresponding surface elements $\mathfrak{M}_1(x) \cap F$ is a function $F \to S_1(M)$, and since any function preserves $\sim$, it follows that the surface elements of F at x and y (both in F, and with $x \sim y$) are neighbours in $S_1(M)$. Furthermore, $y \in \mathfrak{M}_1(x) \cap F$; so y belongs to the surface element of F at x. Thus we see that if F is a surface, the surface elements defined by F at neighbouring points x and y of F are in united position,

$$\mathfrak{M}_1(x) \cap F \approx \mathfrak{M}_1(y) \cap F. \tag{2}$$

This is the motivation for the notion.

Proposition 3 *Let K be a calotte with base point x, and let $y \in K$ be $\sim x$. Then*

$$\mathfrak{M}_1(x) \cap K \approx \mathfrak{M}_1(y) \cap K$$

as elements of $S_1(M)$.

Proof. Let F be any surface with $K = \mathfrak{M}_2(x) \cap F$. Then

$$\mathfrak{M}_1(x) \cap K = \mathfrak{M}_1(x) \cap \mathfrak{M}_2(x) \cap F = \mathfrak{M}_1(x) \cap F,$$

since $\mathfrak{M}_1(x) \subseteq \mathfrak{M}_2(x)$. On the other hand,

$$\mathfrak{M}_1(y) \cap K = \mathfrak{M}_1(y) \cap \mathfrak{M}_2(x) \cap F = \mathfrak{M}_1(y) \cap F,$$

since $\mathfrak{M}_1(y) \subseteq \mathfrak{M}_2(x)$ (by (1)). The result now follows from (2).

3 First order PDEs

By a *first order PDE* on a 3-dimensional manifold M, one understands a 4-dimensional submanifold Ψ of the 5-dimensional manifold $S_1(M)$ of surface elements in M. The *solutions* of Ψ are then the surfaces F in M such that all surface elements of F belong to Ψ.

The method of characteristics is a way to construct such solutions F.

This geometric formulation of the analytic notion of "first order partial differential equation" goes back to Monge, Lie, and other 19th century geometers, cf. classical texts like [7], [1], [9],....

The submanifold $\Psi \subseteq S_1(M)$ is typically given by one "partial differential equation" $\psi(P) = 0$ for a function $\psi : S_1(M) \to R$, so Ψ is generally 4-dimensional since $S_1(M)$ is 5-dimensional. (A 3-dimensional submanifold of $S_1(M)$ is typically given by *two* such partial differential equations; it is an "over-determined" system, and such may have no solutions.)

By a *solution calotte* of Ψ, we mean a calotte K all of whose surface elements belong to Ψ, so for all surface elements P

$$(P \text{ belongs to } K) \text{ implies } (P \in \Psi).$$

A necessary condition that a calotte K at x is a solution calotte is of course that its restriction belongs to Ψ, i.e. that $\mathfrak{M}_1(x) \cap K \in \Psi$. How many solution calottes through P are there, i.e. how many solution calottes are there with restriction P?

We shall in a coordinatized situation ($M = R^3$), for non-degenerate Ψ, prove that the set of such calottes form a 1-dimensional manifold, see the remarks after Theorem 10.

We ask: given $P \in \Psi$, how many surface elements $P' \approx P$ have the property that they belong to *all* these ∞^1 solution calottes extending P? We pose:

Definition 4 *Let Ψ be a PDE, and let $P \approx P'$ be surface elements in Ψ. If P' belongs to* all *solution calottes extending P, we say that P' is a* characteristic *neighbour of P, written $P \approx_\Psi P'$.*

Just as for $\approx$, it can, by analytic means, be proved that $\approx_\Psi$ is a symmetric relation.

- In the following, Ψ is a fixed PDE on M.

Proposition 5 *Let $P \approx_\Psi P'$ with base points x and y, respectively. Assume that F is a solution surface containing P. Then F also contains P'. And $P' = \mathfrak{M}_1(y) \cap F$.*

Proof. Consider the (solution) calotte $K := \mathfrak{M}_2(x) \cap F$. Since K extends P and $P \approx_\Psi P'$, P' belongs to K, so $P' \subseteq K \subseteq F$. The second assertion follows from Proposition 2.

Theorem 6 *If F_1 and F_2 are solution surfaces containing P. Assume that $P \approx_\Psi P'$. Then $\mathfrak{M}_1(y) \cap F_1 = \mathfrak{M}_1(y) \cap F_2$ where y is the base point of P'.*

Proof. For by Proposition 5 $\mathfrak{M}_1(y) \cap F_1 = P' = \mathfrak{M}_1(y) \cap F_2$.

(The conclusion $\mathfrak{M}_1(y) \cap F_1 = \mathfrak{M}_1(y) \cap F_2$ may also be expressed that F_1 and F_2 are *tangent* to each other at y, or *touch each other* at y, see [6].)

If P and P' are characteristic neighbours, and y is the base point of P', then P' can be reconstructed from y and P. For, take any solution calotte K extending P (such calottes do exist - there are in fact ∞^1 of them). Since $P \approx P'$, we have that $y \in P \subseteq K$, and since P' belongs to all such solution calottes by assumption, $P' = \mathfrak{M}_1(y) \cap K$.

The characteristic neighbour relation $\approx_\Psi$ is reflexive and symmetric; in fact, it defines a 1-dimensional distribution on the manifold Ψ, and hence can be integrated into curves in $S_1(P)$. These curves are the classical "characteristic strips" of the PDE Ψ. A *characteristic strip* (w.r. to Ψ) is a strip where neighbouring elements P and P' not only have $P \approx P'$ but $P \approx_\Psi P'$.

A classical method for constructing solution surfaces for Ψ is to build them as unions of characteristic strips; we shall not elaborate on this; see e.g. [1] p. 284.

4 Monge cone

Given a PDE $\Psi \subseteq S_1(M)$ on a manifold M, as above. A point x' which appears as the base point of a characteristic neigbour P' of P may be called a characteristic neighbour *point* of P "in the calotte sense". There is (for $M = R^3$) another, older, notion of characteristic neighbour point of P, going back to Monge, Lagrange, ..., namely, it is a point x' of P, on the *line* along which P is tangent to the "Monge *cone*" at x (where x is the base point of P).

The notions of "line" and "cone" presuppose some linear or affine structure on M, unlike the notions of the previous Section. We shall describe these "affine" notions in synthetic form, assuming that M is linearly isomorphic to R^3. (Points in R^3 are of the form (x_1, x_2, x_3), so we denote such points by $\underline{x}$).

In M is a vector space isomorphic to R^3, the KL axiomatics give that a surface element P with base point $\underline{x}$ gives rise to a unique pointed *plane* V in M, with $\underline{x}$ as base point. If K is a calotte in M with base point $\underline{x}$ and if $(P', \underline{y})$ belongs to K, the pointed plane $(V', \underline{y})$ arising from $(P', \underline{y})$ deserves the name *the tangent plane* to K at $\underline{y}$; and $P' = \mathfrak{M}_1(\underline{y}) \cap V'$.

Therefore, in the classical treatment, with $M = R^3$, the surface elements in R^3 are called *plane* elements, since a surface element at $\underline{x} \in R^3$ may be given by a (unique) *plane* through $\underline{x}$. The plane elements of a PDE Ψ with a given base point $\underline{x}$ have an enveloping surface, which is a cone, called the *Monge cone* at $\underline{x}$; each individual plane element $P \in \Psi$ with base point $\underline{x}$ is tangent to the Monge cone at $\underline{x}$ along a generator of the cone, and this generator $l \subseteq P$ is the *characteristic line* of the plane element. Paraphrasing, we then arrive at the provisional definition that $\underline{y}$ is a characteristic neighbour point (in the "Monge sense") of the plane element $(P, \underline{x}) \in \Psi$ if $\underline{y} \in \mathfrak{M}_1(\underline{x}) \cap l$.

We prove in Section 7 that $\underline{x'}$ is a characteristic neighbour point of P in the "calotte" sense iff it is so in the "Monge" sense.

However, as argued in [4], the relationship between enveloping surfaces and characteristics is that the characteristics are logically prior to the enveloping surface (which is made up of the characteristics). From this conception, it is therefore a detour to define the characteristic lines l in terms of the Monge cones.

Let us be explicit about the synthetic description (following [4] for the notions of characteristics and envelopes) of the characteristic lines and the Monge cones for a given PDE Ψ in R^3. We identify $S_1(M)$ (when $M = R^3$) with the 5-dimensional manifold of pointed planes $P = (\underline{x}, P)$ in R^3, and Ψ with a 4-dimensional submanifold. Let $\Psi(\underline{x})$ be the set of pointed planes with base point $\underline{x}$. Then the characteristic line $l(\underline{x}, P)$ for a given plane element $(\underline{x}, P)$ is

$$\bigcap_{P' \in \Psi(\underline{x}),\ P' \sim P} |P'| \tag{3}$$

and the Monge cone at $\underline{x}$ is the union of these lines, as P ranges over $\Psi(\underline{x})$. (We wrote $|P'|$ to distinguish the plane P' from the *pointed* plane $(\underline{x}, P')$.)

If we return to the case of a general 3-dimensional manifold, where we work with surface elements $(x, P) = \mathfrak{M}_1(x) \cap F$, rather than pointed planes, we are therefore motivated to make the following

Definition 7 *Let Ψ be a PDE on a 3-dimensional manifold M, and let $P \in \Psi$ with base point $\underline{x}$. Then $\underline{y} \in P$ is a* characteristic neighbour point for P *(in the "Monge" sense) if for all $P' \sim P$ in Ψ and with same base point $\underline{x}$ as P, we have $\underline{y} \in P'$.*

Thus, the set of characteristic points of the surface element P at $\underline{x}$ is given by the formula (3), but where now P and P' denote surface elements rather than pointed planes. So in this definition no algebraic structure (like vector space of affine space) is assumed on M.

This may be seen as a rigourous formulation of the description of Lie, [7] p. 510: "*... so hat man im Punkte (x,y,z) die Schnittlinie der Ebenen zweier solcher unmittelbar benachbarte Flächenelemente ... zu suchen ...*" ("... then one must then in the point (x,y,z) look for the intersection line of the planes of two such immediate neighbouring plane elements") (he is talking about two plane elements with same base point (x,y,z)). So instead of intersecting P with "an immediate neighbour" P' in Ψ with base point x, we intersect P with *all* its "immediate" (first order) neigbours in Ψ with base point x.

5 Coordinate calculations

We consider the case where $M = R^3$. A (smooth) function $f : R^2 \to R$ gives rise to a surface F in R^3, namely its graph. Not all surfaces in R^3 come about in this way (they may contain vertical surface elements), but since our notions are local and invariant under diffeomorphisms, it suffices to consider such "graph"-surfaces.

We shall be concerned in particular with linear and quadratic[2] functions, and their graphs. The graphs of the linear functions are planes P. We consider in particular pointed planes $(\underline{x}, P)$, since they give rise to surface elements. A pointed (non-vertical) plane with $\underline{x} = (x,y,z)$ is given by a 5-tuple (x,y,z,p,q) of numbers, namely the graph of the linear function f with $f(x,y) = z$ and with $\partial f \partial x = p$ and $\partial f \partial y = q$, so

$$f(\xi,\eta) = z + p(\xi - x) + q(\eta - y), \tag{4}$$

or writing dx for the "increment" $(\xi - x)$, and similarly dy for $\eta - y$ and dz for $\zeta - z$,

$$f(x+dx, y+dy) = z + p\,dx + q\,dy,$$

or

$$dz = p\,dx + q\,dy.$$

Consider also another the pointed plane with same base point $\underline{x}$, so it given by $(x,y,z,p+\delta p, q+\delta q)$. Then it is elementary arithmetic to prove

Proposition 8 *Consider the pointed planes given by (x,y,z,p,q) and by $(x,y,z,p+\delta p, q+\delta q)$, respectively. Then $(x+dx, y+dy, z+dz)$ belongs to both these planes iff*

$$dz = p\,dx + q\,dy \quad \textit{and} \quad \delta p \cdot dx + \delta q \cdot dy = 0.$$

[2]"linear" is here to be read: polynomial function of degree ≤ 1; similarly, we us "quadratic" for polynomial functions of degree ≤ 2.

For two surface elements P and P' to be in united position, $P \approx P'$, they must first of all be neighbours, $P \sim P'$, so they are of the form

$$(x,y,z,p,q) \quad \text{and} \quad (x+dx,y+dy,z+dz,p+dp,q+dq)$$

respectively, with $(dx,dy,dz,dp,dq) \in D(5)$; and then

$$P \approx P' \quad \text{iff} \quad dz = p\,dx + q\,dy. \tag{5}$$

To prove symmetry of the relation $\approx$, we should from $dz = p\,dx + q\,dy$ deduce that

$$dz = (p+dp)(dx) + (q+dq)(dy);$$

but this follows because $dp \cdot dx = 0$ and $dq \cdot dy = 0$ since $(dx,\ldots,dq) \in D(5)$. (Lie puts it this way, [7] p. 523: "*here, we ignore infinitely small quantities of higher order*"; in our formalism, the "higher order quantities" to be ignored are $dp \cdot dx$ and $dq \cdot dy$; they are both 0.)

An 8-tuple (x,y,z,p,q,r,s,t) defines a quadratic function $f : R^2 \to R$, given by

$$f(\xi,\eta) = z + p(\xi - x) + q(\eta - y) + \tfrac{1}{2}r(\xi - x)^2 + s(\xi - x)(\eta - y) + \tfrac{1}{2}t(\eta - y)^2. \tag{6}$$

The partial derivatives $\partial f/\partial\xi$ and $\partial f/\partial\eta$ are the functions

$$\partial f/\partial\xi(\xi,\eta) = p + r(\xi - x) + s(\eta - y)$$

and

$$\partial f/\partial\eta(\xi,\eta) = q + s(\xi - x) + t(\eta - y)r,$$

respectively, or, writing dx for $\xi - x$ and dy for $\eta - y$,

$$\partial f/\partial\xi`(x+dx,y+dy) = p + r\,dx + s\,dy \tag{7}$$

and

$$\partial f/\partial\eta(x+dx,y+dy) = q + s\,dx + t\,dy \tag{8}$$

Let K be the calotte at (x,y,z) given as the graph of the restriction of this function to $\mathfrak{M}_2(x,y)$. Then every $(dx,dy) \in D(2)$ gives rise to a surface element belonging to K, namely

$$(x+dx,y+dy,z+p\,dx+q\,dy,p+r\,dx+s\,dy,q+s\,dx+t\,dy), \tag{9}$$

and every surface element belonging to K is of this form for unique $(dx,dy) \in D(2)$.

We consider a surface element $P = (x,y,z,p,q)$ and ask for the relation between on the one hand

• calottes $K = (x,y,z,p,q,r,s,t)$ extending P, and

• surface elements $P' = (x+dx,y+dy,z+p\,dx+q\,dy,p+dp,q+dq)$ on the other. (Here, $(dx,dy,dp,dq) \in D(4)$; and surface elements of the described form are automatically in united position with P.)

Proposition 9 *The surface element P' belongs to K iff (r,s,t) is a solution of a certain linear equation system (two equations in three unknowns), namely the linear system with augmented matrix*

$$\left[\begin{array}{ccc|c} dx & dy & & dp \\ & dx & dy & dq \end{array}\right]. \tag{10}$$

Proof. Consider the function f from (6) whose 2-jet at (x,y) has K as graph. Its first partial derivatives at $(x+dx, y+dy)$ are $p+r\,dx+s\,dy$ and $q+s\,dx+t\,dy$, respectively. For P' to belong to K, these partial derivatives have to be $p+dp$ and $q+dq$, respectively. This equation expresses a relation between (r,s,t) and (dx,dy,dp,dq) on the other, which may be rewritten in matrix form as stated.

6 PDEs in coordinates

Now we bring in a PDE Ψ, a 4-dimensional submanifold of the 5-dimensional manifold $S_1(R^3)$ of surface elements in R^3. Our considerations are local, so we may assume that Ψ is given as the zero set of a certain function $\psi : R^5 \to R$, in other words, $(x,y,z,p,q) \in \Psi$ iff $\psi(x,y,z,p,q) = 0$. The graph of a function $f : R^2 \to R$ is then a solution surface iff for all (x,y)

$$\psi(x,y,f(x,y),\tfrac{\partial f}{\partial x}(x,y),\tfrac{\partial f}{\partial y}(x,y)) = 0$$

which is a partial differential equation of order 1.

We proceed to describe the solution calottes for Ψ in analytic terms. A necessary condition that a calotte $K = (x,y,z,p,q,r,s,t)$ is a solution calotte is of course that its restriction (x,y,z,p,q) is in Ψ.

To be a solution calotte means that all surface elements belonging to K are in Ψ. These surface elements are of the form described in (9). So $K = (x,y,z,p,q,r,s,t)$ is a solution calotte if (r,s,t) has the property that

$$\psi(x+dx\, y+dy,\ z+p\,dx+q\,dy,\ p+r\,dx+s\,dy,\ q+s\,dx+t\,dy) = 0 \tag{11}$$

for all $(dx,dy) \in D(2)$. We consider this expression as a function of $(dx,dy) \in D(2)$. We Taylor expand ψ from (x,y,z,p,q) and use $\psi(x,y,z,p,q) = 0$; then we see that (11) equivalent to

$$\begin{aligned}\frac{\partial\psi}{\partial x}\cdot dx + \frac{\partial\psi}{\partial y}\cdot dy + \frac{\partial\psi}{\partial z}\cdot(p\,dx+q\,dy)& \\ +\frac{\partial\psi}{\partial p}\cdot(r\,dx+s\,dy) + \frac{\partial\psi}{\partial q}\cdot(s\,dx+t\,dy) &= 0\end{aligned} \tag{12}$$

where the partial derivatives are to be evaluated in (x,y,z,p,q). As a function of $(dx,dy) \in D(2)$, it is linear, and it vanishes iff it vanishes on all $(dx,0)$ and on all $(0,dy)$. This gives two equations (writing ψ_x for $\partial\psi/\partial x$, evaluated at x,y,z,p,q), and similarly for $\psi_y,\dots\psi_q$)

$$(\psi_x + \psi_z p + \psi_p r + \psi_q s)\; dx = 0,$$

$$(\psi_y + \psi_z q + \psi_p s + \psi_q t)\; dy = 0,$$

and for K to be a solution calotte, they have to hold for all dx in D and for all dy in D.

By the fundamental axiom in SDG, one has the principle of "cancelling universally quantified dx s":

if $a \cdot dx = 0$ *for* all $dx \in D$, *then* $a = 0$.

So if these two equations hold for all dx, and all dys, the two coefficients given by the parentheses are 0. We therefore have

Proposition 10 *Given a surface element $P = (x,y,z,p,q) \in \Psi$ and a calotte $K = (x,y,z,p,q,r,s,t)$ extending P, then K is a solution calotte for Ψ is*

$$\psi_x + \psi_z p + \psi_p r + \psi_q s = 0,$$

and

$$\psi_y + \psi_z q + \psi_p s + \psi_q t = 0.$$

For fixed $P = (x,y,z,p,q)$, the two equations in this Theorem is a linear system, two equations with three unknowns r,s,t and with augmented matrix

$$\left[\begin{array}{ccc|c} \psi_p & \psi_q & & -\psi_x - p\cdot\psi_z \\ & \psi_p & \psi_q & -\psi_y - q\cdot\psi_z \end{array}\right]. \tag{13}$$

Given $P = (x,y,z,p,q) \in \Psi$.

The condition on a neighbour surface element P' that it is belongs to a calotte K is given by a condition on the (r,s,t) of the calotte, namely that it is a solution of the equation system (10) in Proposition 9.

The condition that a calotte extending P is a solution calotte is that (r,s,t) is a solution of the equation system (13) in Proposition 10. To say that P' is a characteristic neighbour of P is therefore to say that whenever (r,s,t) solves (13), it also solves (10).

From the "elementary linear algebra" in the Appendix therefore follows that this is the case iff the augmented matrix in (10) is a scalar multiple of the one in (13) .

We are now assuming that the PDE ψ is non-degenerate (or non-singular), meaning that, at the given (x,y,x,p,q), at least one of ψ_p and ψ_q is invertible, then the rank of the coefficient matrix in (13) is 2, whence it represents a surjective linear map $R^3 \to R^2$; the solution set of the equation system is therefore a 1-dimensional (and affine) subspace of the (r,s,t)-space. So also for a general (sufficiently non-singular) PDE $\Psi \subseteq S_1(M)$, there are ∞^1 solution calottes extending a given $P \in \Psi$. So also for a general (sufficiently non-singular) "abstract" PDE $\Psi \subseteq S_1(M)$, there are ∞^1 solution calottes extending a given $P \in \Psi$.

Therefore we have

Theorem 11 *Assume $P = (x,y,z,p,q)$ is in Ψ. For $P' = (x+dx, y+dy, z+p\,dx+q\,dy, p+dp, q+dq)$ to be a characteristic neighbour element, it is necessary and sufficient that there exists a scalar λ such that*

$$(dx,dy,dz,dp,dq) = \lambda \cdot (\psi_p, \psi_q, p\cdot\psi_p + q\cdot\psi_q, -\psi_x - p\cdot\psi_z, -\psi_y - q\cdot\psi_z), \quad (14)$$

or equivalently, that

$$(dx,dy,dp,dq) = \lambda \cdot (\psi_p, \psi_q, -\psi_x - p\cdot\psi_z, -\psi_y - q\cdot\psi_z). \quad (15)$$

Here, ψ_p denotes $\partial\psi/\partial p$ evaluated at $P = (x,y,z,p,q)$, and simlarly for ψ_q, ψ_x etc. Note that our assumption that at least one of ψ_p and ψ_q is invertible implies that the scalar λ is uniquely determined.

From the Theorem follows in particular that for $(x+dx, y+dy, z+p\,dx+q\,dy)$ to be a characteristic neighbour point of P (in the calotte sense), it is necessary that

$$(dx,dy) = \lambda \cdot (\psi_p, \psi_q). \quad (16)$$

In fact, it is also sufficient, since the relevant dp and dq then can be reconstructed from λ and the partial derivatives of ψ by (15).

7 Differential equation for Monge characteristics

We consider the surface element $P = (x,y,z,p,q)$ in Ψ, so $\psi(x,y,z,p,q) = 0$. A neighbour surface element with same base point is of the form $(x,y,z,p+\delta p, q+\delta q)$ with $(\delta p, \delta q) \in D(2)$, and this element is in Ψ if $\psi(x,y,z,p+\delta p, q+\delta q) = 0$; by Taylor expansion, and using $\psi(x,y,z,p,q) = 0$, this is equivalent to

$$(\partial\psi/\partial p)\cdot\delta p + (\partial\psi/\partial q)\cdot\delta q = 0, \quad (17)$$

where the partial derivatives are to be evaluated at (x,y,z,p,q).

A point in P is of the form $(x+dx, y+dy, z+p\,dx+q\,dy)$ with $(dx,dy) \in D(2)$, and this point is in the surface element $(x,y,z,p+\delta p, q+\delta q)$ iff $p\,dx+q\,dy = (p+\delta p)\cdot dx + (q+\delta q)\cdot dy$, that is, iff

$$dx \cdot \delta p + dy \cdot \delta q = 0. \tag{18}$$

So to say that $(x+dx, y+dy)$ is a Monge-characteristic neighbour of P is to say that all $(\delta p, \delta q) \in D(2)$ which satisfy (17) also satisfy (18). Assuming, as before, that $\partial\psi/\partial p$ and $\partial\psi/\partial q$ do not vanish simultaneously, this property is equivalent to: (dx,dy) is of the form $\lambda \cdot (\partial\psi/\partial p, \partial\psi/\partial q)$. Thus, $(x+dx, y+dy, z+p\,dx+q\,dy)$ is a Monge-characteristic neighbour of $P = (x,y,z,p,q)$ iff

$$(dx,dy) = \lambda \cdot (\partial\psi/\partial p, \partial\psi/\partial q).$$

We see that this is just the equation (15) for characteristic neighbour points in the calotte sense. We conclude that the two notions of "characteristic neighbour point" agree. This makes sense, since we noted in the end of Section 3 that P' can be reconstructed from P and the base point y of P'.

.

Appendix

Basically, when working in coordinates, the method of SDG is the method of Taylor expansions of functions in several variables, but together with an explicitly formulated theory of which terms (quantities) "can be ignored", meaning that they are $= 0$. This is not a quantitative question of "being infinitely small", but a qualitative one of being nilpotent of a certain order.

These notions can be formulated in exact terms in terms of basic ring commutative ring R serving as a number line. Let us list some basic constructs growing out of any commutative ring R, whose elements we call "numbers". First of all, we have the subset $D \subseteq R$ of numbers $x \in R$ with $x^2 = 0$; the elements of D we call *first order infinitesimals*. We write $x \sim y$ if $(x-y) \in D$.

More generally, $D(n) \subseteq R^n$ is

$$D(n) := \{(x_1, \ldots, x_n \in R^n \mid dx_i dx_j\},$$

(for all $i, j = 1, ..., n$), not to be confused with $D^n \subseteq R^n$ which is larger: only $dx_i^2 = 0$ is required, for $i = 1, ..., n$.

The basic (KL-) axiom of SDG says that $D(n)$ classifies linear maps $R^n \to R$, meaning that any $f : D(n) \to R$ extends uniquely to a linear map $R^n \to R$. (Here "linear" means "polynomial of degree ≤ 1" or "affine"). For $n = 1$, the uniqueness assertion here implies the principle of cancelling universally quantified ds: if $a \cdot d = 0$ for all $d \in D$, then $a = 0$.

The first order neighbour relation $\sim$ in R^n is defined by $\underline{x} \sim \underline{y}$ if $(\underline{x} - \underline{y}) \in D(n)$.

We have similarly $D_2 \subseteq R$, consisting of the numbers $x \in R$ with $x^3 = 0$, called the *second order infinitesimals*. More generally $D_2(n) \subseteq R^n$ is defined by $\ldots$, and the resulting second order neighbour relation $\sim_2$ defined by $\underline{x} \sim_2 \underline{y}$ if $(\underline{x} - \underline{y}) \in D_2(n)$.

Recall that for $\underline{x} \in R^n$, the subsets $\mathfrak{M}(\underline{x})$ and $\mathfrak{M}_2(\underline{x})$ were described in the Introduction for general M, e.g. $\mathfrak{M}(\underline{x}) = \{\underline{x}' \mid \underline{x} \sim \underline{x}'\}$. Thus in R^n, $\mathfrak{M}(\underline{0}) = D(n)$. The basic axiomatics of SDG may be expressed by saying that a function $f : \mathfrak{M}(\underline{x}) \to R$ extends uniquely to a (globally defined) polynomial function $f : R^n \to R$ of degree ≤ 1 (an affine function). The coefficients of this affine function are the first order partial derivatives of f at $\underline{x}$, (and the constant term $f(\underline{x})$). The graph F of this function is the tangent plane at the point $(\underline{x}, f(\underline{x}) \in R^{n+1}$. In turn, this tangent plane (together with the point $(\underline{x}, f(\underline{x}))$ in it. For $n = 2$, this contains the same information as the surface element of the surface (the graph) at this point.

Similarly, a function $f : \mathfrak{M}_2(\underline{x}) \to R$ extends uniquely to a polynomial function $R^n \to R$ of degree ≤ 2, whose coefficients are the partial derivatives of order ≤ 2 of f.

Any linear map $R \to R$ is multiplication by a unique $\lambda \in R$. From this follows, for any vector space A:

Proposition 12 *Let $p : A \to R$ be a surjective linear map, and let $q : A \to R$ be any linear map. If the kernel of p is contained in the kernel of q, then $q = \lambda \cdot p$ for a unique $\lambda \in R$.*

Proof. Contemplate the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & Ker(p) \arrow[r] \arrow[d, "incl"'] & A \arrow[r, "p"] \arrow[d, "id"] & R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Ker(q) \arrow[r] & A \arrow[r, "q"'] & R &
\end{tikzcd} .$$

The right hand vertical map exists by exactness of top row, and is multiplication by a unique scalar.

Let $p : A \to R$ be a linear map, and let $r \in R$. If $p(x_0) = r$, then the solution set of the equation $p(x) = r$ is the coset $x_0 + Ker(p)$. As a Corollary of the above Propostion, we then have

Proposition 13 *Let $p : A \to R$ be a surjective linear map, and $q : A \to R$ any linear map. Let $r,s \in R$. If the solution set of $p(x) = r$ is contained in the solution set of $q(x) = s$, then there is a unique $\lambda \in R$ so that $q = \lambda \cdot p$ and $s = \lambda \cdot r$.*

Proof. Take some $x_0 \in A$ such that $p(x_0) = r$, using p surjective. The assumption then implies that $q(x_0) = s$. The solution sets of the two equations are, respectively, $x_0 + \ker(p)$, and $x_0 + \ker(q)$, and the assumed inclusion relation then clearly implies $\ker(p) \subseteq \ker(q)$. By the previous Proposition, there is a unique $\lambda \in R$ with $q = \lambda \cdot p$. We then have

$$\lambda \cdot r = \lambda \cdot p(x_0) = q(x_0) = s.$$

Proposition 14 *Consider two linear equation systems given by the augmented matrices*

$$\left[\begin{array}{ccc|c} p_1 & p_2 & & r_1 \\ & p_1 & p_2 & r_2 \end{array}\right] \tag{19}$$

and

$$\left[\begin{array}{ccc|c} q_1 & q_2 & & s_1 \\ & q_1 & q_2 & s_2 \end{array}\right] \tag{20}$$

respectively, and assume that at least one of the p_is is invertible. Assume that the solution set of the first is contained in the solution set of the second. Then there exists a unique $\lambda \in R$ with

$$\lambda \cdot (q_1, q_2, s_1, s_2) = (p_1, p_2, r_1, r_2).$$

Proof. Without loss of generality, we may assume that p_2 is invertible. Assume (x,y) solves $p_1x + p_2y = r_1$; then there is a (unique) z so that (x,y,z) solves the system (19). Hence by assumption, it solves (20), and so (x,y) solves $q_1x + q_2y = s_1$. From Proposition 13 then follows that there exists a λ such that $q_1 = \lambda p_1$, $q_2 = \lambda p_2$ and $s_1 = \lambda r_1$. We prove that also $s_2 = \lambda r_2$: with the unique z already considered, we have $p_1y + p_2z = r_2$; multiplying this equation by λ, we get $q_1y + q_2z = \lambda r_2$, but the left hand side here is s_2 since (x,y,z) solves (20). This proves the Proposition.

References

[1] Felix Klein, Höhere Geometrie, 3rd Edition, Springer 1926.

[2] Anders Kock, Synthetic Differential Geometry, Cambridge University Press 1981 (2nd Edition 2006).

[3] Anders Kock, First neighbourhood od the diagonal, and geometric distributions, Univ. Iagellonicae Acta Math. 41 (2003), 307-318.

[4] Anders Kock, Envelopes - notion and definiteness, Beiträge zur Algebra und Geometrie 48 (2007), 345-350.

[5] Anders Kock, Synthetic Geometry of Manifolds, Cambridge Tracts in Mathematics 180, Cambridge University Press 2010.

[6] Anders Kock, Metric spaces and SDG,Theory and Applications of Categories 32 (2017), 803-822.

[7] Sophus Lie, Geometrie der Berührungstransformationen, Leipzig 1896 (Reprint by Chelsea Publ. Co. 1977).

[8] A. Kumpera and D. Spencer, Lie Equations, Vol. I, Annals of Math. Studies No. 73 (1972).

[9] W.W. Stepanow, Lehrbuch der Differentialgleichungen, VEB Deutscher Verlag der Wissenschaften, Berlin 1982.

Dept. of Math. University of Aarhus, November 2022

kock@math.au.dk