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GROUP VALUED DIFFERENTIAL FORMS REVISITED

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Group valued differential forms revisited

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For Jim Stasheff

Abstract We study the relationship between combinatorial group valued differential forms, and classical differential forms with values in the corresponding Lie algebra. In particular, we compare simplicial coboundary and exterior derivative for 1-forms. The results represent strengthenings of results obtained by the author in 1982.

1 Introduction

The present paper is partly expository, since the main result – the relationship between the theory of combinatorial group valued differential forms, and the corresponding classical Lie algebra valued differential forms –, in some sense is contained already in my [5] (notably in the form of Theorem 6 below, where the formula is the same as the formula in Theorem 5.4 in [5]). However, a senior colleague of mine pointed out that the correspondence between the two kinds of forms was not made fully explicit there, and urged me to make it so; in the process of doing this, I was forced to close some theoretical holes.

The complete correspondence between the two kinds of forms has two passages:

group valued combinatorial forms $\to$ vector space valued combinatorial forms $\to$ vector space valued classical forms

The second passage was not really made explicit in [5] or [7], – in fact, in the latter, we use the term "classical differential form" for combinatorial forms as soon as the value group is a vector space. For instance, the "classical curvature" in loc.cit. is still a *combinatorial* form.

The present paper aims at presenting the theory more completely. Therefore, it also contains some repetitions of material already published. But the present exposition has a wider and more uniform level of generality, where [5] deals with two ad hoc cases, the most important being where the value group G is a finite dimensional manifold, i.e. a Lie group.

Also, we utilize some tools that were not available at the time of [5], notably the theory of affine combinations of mutual neighbour points in a manifold. The log function, as introduced in [10], depends on this notion; and this log-function is needed for establishing a coordinate free comparison between combinatorial and classical differential forms, which is one new feature in the present work.

Another new feature of foundational nature is the introduction of a (first-order) neighbour relation in an *arbitrary* object G, – where [4] only has this relation for finite dimensional manifolds.

We assume some basic familiarity with the technique of Synthetic Differential Geometry (SDG), but summarize in the Appendix some particular concepts. The basic object is a commutative ring R, the "number line", and, viewed as an R-module, it is assumed to satisfy the KL axiom scheme, as partly expounded in the Appendix below.

In this context of SDG, we consider a class of groups which are suitable for being value groups for differential 1-forms, and their coboundary. In [5], we considered groups that were either manifolds (i.e. Lie groups), or groups which were groups of diffeomorphisms of manifolds. The theory presented here comprises these two cases. It is more general, and hence the proofs, hopefully, clearer.

The groups considered here are subgroups G of associative unitary algebras A (i.e. $G \subseteq A$ is a subgroup of the multiplicative monoid of A). The only further requirement is that the underlying vector space (R-module) of A is a "Euclidean" R-module, i.e. satisfies the vector form of the general KL axiom scheme (see Appendix). (Also, in order to to be able to define a Lie algebra structure on the tangent space $T_e(G)$, we need that G satisfies a certain "micro-linearity" condition; this is a very mild condition. Similarly for the vector space structure of $T_e(G)$.) However, the manifolds M on which the differential forms live, are assumed finite dimensional, so that the theory of combinatorial differential forms on M (in the sense of [5], [4] I.18) makes sense.

One may think of A as an *enveloping algebra*, not only of $T_e(G)$ (and not necessarily the *universal* enveloping algebra), but also an enveloping algebra of G itself. The algebra A is auxilary, and ultimately, one wants (and we do prove) results that are independent of A.

In this axiomatic framework, we shall be discussing the relationship between the Lie algebra $\mathfrak{g}$ of G, on the one hand, and A, on the other. (Recall that this Lie algebra $\mathfrak{g}$ is by definition the tangent space $T_e(G)$; we shall below review its Lie algebra structure, as it may be constructed in the context of SDG.) In Section 9, we give the comparison with the classical theory of differential forms with values in the Lie algebra of a Lie group. The comparison implies the classical Maurer-Cartan formula.

(We shall use the phrase "vector space" as synonymous with "R-module", and the phrase "Euclidean vector space" as synonymous with "R-module satisfying the general KL axiom", cf. the Appendix below. "Linear" means "R-linear", and similarly for "multilinear".)

2 The neighbour relation, and infinitesimal simplices

In the context of SDG, any finite dimensional manifold M carries a reflexive symmetric relation, the neighbour relation $x \sim y$; and any (smooth) map between manifolds preserves this relation. For $M = R^n$, $x \sim y$ iff $x - y \in D(n)$. Here, $D(n) \subseteq R^n$ is one of the the standard infinitesimal objects of SDG:

$$D(n) := \{(d_1, \ldots, d_n) \in R^n \mid d_i \cdot d_j = 0 \quad \forall i, j = 1, \ldots, n\}$$

in particular $d_i^2 = 0$ for all i. It may be characterized in a coordinate free way by the condition: $\underline{d} \in D(n)$ iff for every R-module W and for every bilinear $\Phi : R^n \times R^n \to W$, we have $\Phi(\underline{d}, \underline{d}) = 0$.

We may extend the neighbour notion to arbitrary objects A, by declaring $a_1 \sim a_2$ if there exists an open[1] subset U of some R^n, a map $f : U \to A$, and some $x_1 \sim x_2$ in U with $f(x_1) = a_1, f(x_2) = a_2$. We say that this U, f, x_1 and x_2 *witness* that $a_1 \sim a_2$. If A is itself a finite dimensional manifold, this new relation agrees with the original $\sim$ on A. Also clearly, any map $g : A \to B$ preserves the neighbour relation $\sim$ thus introduced.

In a finite dimensional manifold M, an *infinitesimal k-simplex* is (cf. [4]) by definition a $k+1$-tuple $(x_0, x_1, \ldots, x_k)$ of mutual neighbours, $x_i \sim x_j$ for all $i, j = 0, \ldots, k$. We would like also to have a notion of infinitesimal k-simplex in an arbitrary object A; it should likewise be a $k+1$-tuple of mutual neighbour points $(a_0, \ldots, a_k)$, but we need furthermore to assume the existence of a "uniform" witness $f : U \to A$ for the required neigbourhood relations. So explicitly, to say that $k+1$-tuple $(a_0, \ldots, a_k) \in A^{k+1}$ is an infinitesimal k-simplex in A is to say that there exists some open U in some R^n, a map $f : U \to A$, and an infinitesimal k-simplex $(x_0, \ldots, x_k)$ in U with $f(x_i) = a_i$ for $i = 0, \ldots, k$.

Here, we shall mainly be interested in the notion of infinitesimal k-simplex for $k = 1$ and $k = 2$. A 1-simplex is just a pair of neighbour points, as described above, ultimately in terms of some $D(n)$. To describe the infinitesimal 2-simplices (x, y, z) in R^n, we shall from [4] I.16 recall another "infinitesimal" object $\widetilde{D}(2, n) \subseteq R^n \times R^n$ (in [7] denoted $\Lambda^2 D(n)$); it is the set of $2 \times n$ matrices $[x_{ij}]$ with entries from R, such that both rows are in $D(n)$ and furthermore $x_{1j}x_{2j'} + x_{1j'}x_{2j} = 0$ for all $j, j' = 1, \ldots, n$. In R^n, the points $0, x, y$ form an infinitesimal 2-simplex iff $(x, y) \in \widetilde{D}(2, n)$. In spite of this coordinate dependent description, the $\widetilde{D}$ makes "invariant" sense: for any n-dimensional vector spaces V (i.e. $V \cong R^n$), one may define $\widetilde{D}(2, V)$. (There are also "higher" $\widetilde{D}(k, V)$, suitable for describing infinitesimal k-simplices for $k > 2$, see [4].) The remarkable and crucial property of $\widetilde{D}(2, V)$ is that the general KL axiom (see Appendix) for an R-module A implies that *if a map $f : \widetilde{D}(2, V) \to A$ has $f(0, \underline{d}) = f(\underline{d}, 0) = 0$ for all $\underline{d} \in D(n)$, then f extends uniquely to a bilinear and alternating map $V \times V \to A$.* (There are similar properties for $\widetilde{D}(k, V)$.) – All this may be found in [4] I.16 (for the case $V = R^n$, $A = R$), or in [13].

Let us note the following consequence; V still denotes a finite dimensional vector space, and A a Euclidean vector space.

Proposition 1 *If $f : V \times V \to A$ is bilinear, and $(x, y) \in \widetilde{D}(2, V)$, then $f(x, y) = -f(y, x)$.*

[1]the notion of "open", and the notion of "manifold", are discussed in the Appendix.

Proof. We have $f(0, \underline{d}) = f(\underline{d}, 0) = 0$ for all $\underline{d} \in V$, by bilinearity of f. The restriction of f to $\widetilde{D}(2, V) \subseteq V \times V$ extends, by the "remarkable property" mentioned above to an *alternating* bilinear map $F : V \times V \to A$; and $F(x, y) = f(x, y)$, and similarly for (y, x), since (x, y) (and hence (y, x)) belongs to $\widetilde{D}(2, V)$. Thus

$$f(x, y) = F(x, y) = -F(y, x) = -f(y, x).$$

We sometimes express the conclusion of this Proposition by saying: if $(x, y) \in \widetilde{D}(2, V)$, then any bilinear map *behaves* on (x, y) *as if* it were alternating.

One consequence of Euclideanness of A is that if $0_A \sim a$ in A, then there is, for some n, a *linear* map $F : R^n \to A$ and a $\underline{d} \in D(n)$ with $F(\underline{d}) = a$. For, $0_A \sim a$ is by definition witnessed by some $f : U \to A$ (U an open subset of some R^n) and some $u \sim v$ in U with $f(u) = 0_A$ and $f(v) = a$. Without loss of generality, we may assume that $u = 0 \in R^n$, and hence $v \in D(n)$. Now the restriction of f to $D(n)$ (which is a subset of U, since $0 \in U$ and U is open) is a map $D(n) \to A$ taking 0 to 0_A, and thus by Euclideanness of A, it extends (uniquely) to a linear map $F : R^n \to A$, and this F, and $\underline{d} = v$, proves the assertion.

In the same spirit, we have that if $0_A, a$, and b form an infinitesimal 2-simplex in A, then there is, for some n, a linear map $F : R^n \to A$, and a $(\underline{d}_1, \underline{d}_2) \in \widetilde{D}(2, n) \subseteq R^n \times R^n$ with $F(\underline{d}_1) = a$ and $F(\underline{d}_2) = b$.

Note that if $\tau : D \to A$, then $\tau(d) \sim \tau(0)$ for any $d \in D$; for, τ extends by KL (uniquely) to an affine map $T : R \to A$, and T, together with $d \sim 0$ witnesses $\tau(d) \sim \tau(0)$. Also, if G is a finite dimensional manifold, and $\tau : D \to G$, one may prove $\tau(d) \sim \tau(0)$. But for a general object G, and $\tau : D \to G$, there is no reason why $\tau(d) \sim \tau(0)$ in G, unless τ may be extended to some open $U \supseteq D$. Also, if $G \subset A$, and g_1 and g_2 are in G, then from $g_1 \sim g_2$ in A does not follow that $g_1 \sim g_2$ in G.

3 Associative algebras

We shall now consider an associative algebra $(A, \cdot)$, whose underlying vector space is Euclidean (but not necessarily finite dimensional). The multiplicative unit is denoted e, the additive unit (zero vector) is denoted 0_A or just 0.

Proposition 2 *Let $a \sim 0_A$ in A. Then $a \cdot a = 0_A$. More generally, for any $u \in A$, $a \cdot u \cdot a = 0_A$.*

Proof. There exists, as we observed, some linear map $F : R^n \to A$ and some $y \in D(n)$ witnessing $a \sim 0_A$ (so $F(y) = a$). Since the multiplication $\cdot$ on A is bilinear, we therefore have a bilinear map

$$R^n \times R^n \xrightarrow{F \times F} A \times A \xrightarrow{\cdot} A, \tag{1}$$

and $a \cdot a = F(y) \cdot F(y)$. But since $y \in D(n)$, the bilinear map (1) kills (y, y), hence $a \cdot a = 0_A$. For the more general assertion in the Proposition, one replaces the multiplication map $A \times A \to A$ in (1) by the, likewise bilinear, map $A \times A \to A$ given by $(b, c) \mapsto b \cdot u \cdot c$.

Proposition 3 *Let $a \sim 0_A$. Then $e + a$ has a multiplicative two-sided inverse, namely $e - a$.*

Proof. This is an immediate consequence of Proposition 2; for

$$(e + a) \cdot (e - a) = e + a - a - a \cdot a = e,$$

since $a \cdot a = 0_A$ by Proposition 2. Similary, $(e - a) \cdot (e + a) = e$.

Much in the same spirit, we have

Proposition 4 *Let 0_A, a_1 and a_2 form an infinitesimal 2-simplex in A. Then $a_1 \cdot a_2 = -a_2 \cdot a_1$.*

Proof. As in the proof of Proposition 2, we may assume that the simplex is witnessed by a linear $F : R^n \to A$ and an infinitesimal 2-simplex of the form $(0, x_1, x_2)$ with $(x_1, x_2) \in \widetilde{D}(2, n) \subseteq R^n \times R^n$. Now apply Proposition 1 to the bilinear map $(x, y) \mapsto F(x) \cdot F(y)$.

4 Lie structure

We now consider the case where there is given a subgroup G of the multiplicative monoid of A (as in the previous section, A is assumed to be an associative unitary algebra whose underlying vector space is Euclidean.)

Example 1. Let $A = gl(n, R)$, the algebra of $n \times n$ matrices with entries from R; let $G = GL(n, R) \subseteq gl(n, R)$ be the group of invertible matrices. This is a particularly simple example, since here one can prove that the Lie algebra $\mathfrak{g}$ of G is $gl(n, R) = A$ (with algebraic commutator $xy - yx$ as Lie bracket).

Example 2. We take $A = gl(n, R)$ like Example 1, but with G the group $SL(n, R)$ of matrices of determinant 1. Its Lie algebra consists of matrices of trace 0 (see e.g. [3] I.7); they do not form a subalgebra of $gl(n, R)$, "algebra" meaning "associative algebra".

Both these examples have A finite dimensional (hence Euclidean), and with G a manifold.

Example 3. (cf. [2] II.4.5; see also [4] I.12 and in particular [6]). Let G be a group (a Lie group, say). Let A be the vector space "of distributions on G with compact support"; it can in the present synthetic context be construed as the vector space $Lin_R(R^G, R)$ of linear maps $R^G \to R$. It is Euclidean. Multiplication is convolution of distributions (using the multiplication of G). Every $g \in G$ gives rise to a punctual distribution, namely the Dirac distribution δ_g at g. (To make sure that this is an example, one needs that to prove that the map $g \mapsto \delta_g$ is injective, which is probably not generally possible, on the meager axiomtic basis we are using here.)

Example 4. Let M be a manifold. We have the vector space R^M of functions on M. Let A be the vector space of R-linear maps $R^M \to R^M$; it becomes an associative unitary algebra by taking composition of functions as multiplication.

The group G of invertible maps $M \to M$ (= the group of diffeomorphisms) is a subgroup of the multiplicative monoid of this algebra: to $f : M \to M$, associate the linear map $R^M \to R^M$ "precompose by f". (To make sure that this is an example, one needs some injectivity, as in the previous example.)

(I don't know whether the A = Universal Enveloping Algebra of the Lie algebra $T_e(G)$ will work; is it a KL vector space ? and does it contain G ?)

Let $G \subseteq A$ as above. We have three binary operations which are closely related, in a sense we shall make explicit; namely

• 1) the algebraic commutator on A,

$$[a, b] := a \cdot b - b \cdot a,$$

• 2) the group commutator on G

$$\{x, y\} := x \cdot y \cdot x^{-1} \cdot y^{-1},$$

• 3) and finally the Lie bracket $[[-, -]]$ on $T_e(G)$ – whose synthetic construction we shall recall below, and for which we need to assume that G has a certain "micro-linearity" property, which we shall likewise recall.

Recall that a *tangent vector* to an object M at $x \in M$ is a map $\tau : D \to M$ with $\tau(0) = x$. The set of tangent vectors to m at x is denoted $T_x(M)$.

Since A is assumed a Euclidean vector space, any tangent vector τ to A, at $a \in A$, say, can be written in the form $\tau(d) = a + d \cdot v$, for some unique $v \in A$. This element v is called the *principal part* of the tangent vector τ.

We consider now in particular $T_e(G)$, the set of tangent vectors to G at e – traditionally $T_e(G)$ is denoted $\mathfrak{g}$. We may consider such a tangent vector as a tangent vector τ to A at e; as such, it therefore has a principal part, which we shall call $L(\tau)$. We thus get a map

$$L : T_e(G) \to A;$$

$L(\tau)$ is explicitly given by the equation

$$\tau(d) = e + d \cdot L(\tau) \quad \forall d \in D. \tag{2}$$

An equivalent description is obtained in terms of the map $l : G \to A$, defined as follows:

$$l(g) = g - e \text{ for } g \in G.$$

(One may think of l as a kind of logarithm function.) Then L may be characterized by

$$d \cdot L(\tau) = l(\tau(d)) \quad \forall d \in D.$$

The map $L : T_e(G) \to A$ is clearly injective.

From Proposition 3 we have that if $a \sim 0_A$, then $e + a$ is invertible, $(e + a)^{-1} = e - a$. If further $e + a \in G$, then so is $e - a$, since G by assumption is stable under multiplicative inversion in A. If we have that both $a \sim 0_A$ and $b \sim 0_A$, with $e+a$ and $e+b$ in G, we may therefore form their group theoretic commutator in G,

$$\{e+a, e+b\} = (e+a) \cdot (e+b) \cdot (e+a)^{-1} \cdot (e+b)^{-1};$$

using $(e+a)^{-1} = e-a$ and similarly for b, we rewrite this as a product of four two-term factors:

$$\{e+a, e+b\} = (e+a) \cdot (e+b) \cdot (e-a) \cdot (e-b).$$

Multiplying out by the distributive law, we get 16 terms, but most of them (11 in fact) contain either a factor $\pm a$ twice, or a factor $\pm b$ twice, and so vanish, by Proposition 2 (second clause). Two of the remaining five terms are of the form $\pm a \cdot b$ and cancel each other, and this leaves three terms, $e + a \cdot b - b \cdot a$. So we have proved

Proposition 5 *If a and b are neighbours of 0_A, and $e+a \in G$, $e+b \in G$, then*

$$\{e+a, e+b\} = e + [a, b] \in G,$$

where $[a, b]$ denotes the usual "algebraic" commutator $a \cdot b - b \cdot a$.

Equivalently, if $g, h \in G$ and are $\sim e$ in A (in particular, if they are $\sim e$ in G), then

$$l(\{g, h\}) = [l(g), l(h)].$$

We shall now recall the synthetic description of a Lie bracket $[[-,-]]$ on $\mathfrak{g} = T_e(G)$, following [17] (or see [4] I.9); we need to assume a suitable "microlinearity condition" for G, namely: "if a map $\tau : D \times D \to G$ satisfies $\tau(d, 0) = \tau(0, 0) = \tau(0, d)$ for all $d \in D$, then there exists a unique $t : D \to G$ with $\tau(d_1, d_2) = t(d_1 \cdot d_2)$". The "surrounding" A automatically satisfies this, being Euclidean; see [15] 2.3 for a proof of this for the case $A = R$. (The microlinearity condition is related to the notion from [1] of an "object which universally reverses infinitesimal pushouts".)

Given ξ and η in $T_e(G)$. For d_1 and d_2 in D, we have the group theoretic commutator $\{\xi(d_1), \eta(d_2)\} \in G$. It takes value e whenever either d_1 or d_2 is 0. By the microlinearity condition assumed for G, it follows that there is a unique tangent vector t of G at e with the property that $t(d_1 \cdot d_2) = \{\xi(d_1), \eta(d_2)\}$ for all $(d_1, d_2) \in D \times D$. We denote this t by the symbol $[[\xi, \eta]]$, so the characterizing equation of $[[\xi, \eta]]$ (the "Lie bracket") is that

$$[[\xi, \eta]](d_1 \cdot d_2) = \{\xi(d_1), \eta(d_2)\}.$$

Theorem 1 *The map $L : T_e(G) \to A$ takes the Lie bracket $[[-,-]]$ on $T_e(G)$ to the algebraic commutator $[-,-]$ on A.*

Proof. Let ξ and η belong to $T_e(G)$. To prove $L([[\xi, \eta]]) = [L(\xi), L(\eta)]$, it suffices to prove for all $(d_1, d_2) \in D \times D$ that

$$d_1 \cdot d_2 \cdot L([[\xi, \eta]]) = d_1 \cdot d_2 \cdot [L(\xi), L(\eta)]$$

for all d_1 and d_2 in D (this is just a matter of cancelling the two universally quantified d_is, one at a time, see Appendix for this cancellation principle). We calculate

$$\begin{aligned} d_1 \cdot d_2 \cdot L([[\xi, \eta]]) &= [[\xi, \eta]](d_1 \cdot d_2) - e \\ &= \{\xi(d_1), \eta(d_2)\} - e \end{aligned}$$

(by the definition of $[[-,-]]$ in terms of $\{-,-\}$)

$$= [\xi(d_1) - e, \eta(d_2) - e]$$

(by Proposition 5)

$$= [d_1 \cdot L(\xi), d_2 \cdot L(\eta)]$$

which equals the right hand side of the desired equation, by the bilinearity of the algebraic commutator $[-,-]$. This proves the Theorem.

There is also a structure of vector space on $T_e(G)$, again using a suitable microlinearity condition on G; it can also be described in terms of affine combinations in G. The map $L : T_e(G) \to A$ likewise preserves the vector space structure; for, addition of tangent vectors is here tantamount to addition of their "principal parts", as formed in A. We omit details, which may be found in [4] I.7.

5 Combinatorial differential forms

Let us remind the reader how the notion of infinitesimal simplex gives rise to the notion of combinatorial differential form[2], [4] I.18, (and [1] in a somewhat different context). If $(G, \cdot)$ is a group with neutral element e, a combinatorial differential G-valued k-form ω on a finite dimensional manifold M is a law which to each infinitesimal k-simplex $(x_0, \ldots, x_k)$ in M associates an element $\omega(x_0, \ldots, x_k) \in G$, subject to the sole requirement that the value is e if two of the x_is agree, $x_i = x_j$ for some $i \neq j$. [3]

In the following, the word "form", without further qualifications, means "combinatorial differential form". We are mainly interested in the case of k-forms for $k = 0$, 1, 2. If the value group is the additive group $(V, +)$ of a Euclidean vector space, we shall prove that such a differential form is *alternating*, i.e. that it changes sign when x_i is interchanged with x_j. This is stronger than the usual 'alternating' property of classical differential k-forms, since the latter refers to the symmetric group in k letters, whereas the alternating property for combinatorial k-forms refers to the symmetric group in $k+1$ letters. In particular, the alternating property for combinatorial 1-forms ω, with values in $(V, +)$, says that for $x \sim y$ in M

$$\omega(x, y) = -\omega(y, x) : \tag{3}$$

Proposition 6 *Let ω be a combinatorial differential k-form on a finite dimensional manifold M with values in a Euclidean vector space A. Then ω is alternating.*

[2]This viewpoint was advocated in the late 1970s by Bkouche and Joyal (unpublished).

[3]This requirement can under suitable conditions even be further weakened, see the Remark in Section 8.

Proof. For simplicity of notation, we do the case $k = 2$ only. Since the question is "local" in M, which is assumed to be a manifold, we may as well assume that M is an open subset of a finite dimensional vector space, say $M \subseteq R^n$. For each $x \in M$, consider the map $\Omega(x; -, -) : \widetilde{D}(2, n) \to A$ given by $(\underline{d}_1, \underline{d}_2) \mapsto \omega(x, x + \underline{d}_1, x + \underline{d}_2)$. Since $\omega(x, x, z) = 0$, the map $\Omega(x; 0, -)$ is constant zero, and similarly for $\Omega(x; -, 0)$. Since A is Euclidean, $\Omega(x; --)$ therefore extends uniquely to a bilinear alternating map, likewise denoted $\Omega(x; -, -) : R^n \times R^n \to A$. Jointly, these maps $\Omega(x; -, -)$ define a map $\Omega(-; -, -) : M \times R^n \times R^n \to A$, which is bilinear and alternating in the arguments after the semicolon. So for any infinitesimal 2-simplex (x, y, z),

$$\omega(x, y, z) = \Omega(x; y - x, z - x).$$

This already proves the "alternating" property of $\omega(x, y, z)$ in so far as y and z are concerned. To prove the remaining alternatingness conditions, it suffices therefore to prove that the value changes sign when x and y are interchanged, $\omega(x, y, z) = -\omega(y, x, z)$. To prove this, we need a Taylor expansion of $\Omega(x; y, z)$ in the non-linear variable in front of the semicolon (which is possible, since A is Euclidean/KL, see Appendix). Thus

$$\omega(y, x, z) = \Omega(y; x - y, z - y) = \Omega(x; x - y, z - y) + D\Omega(x; x - y, z - y, y - x),$$

where $D\Omega(x; u, v, w)$ denotes the directional derivative of $\Omega(x; u, v)$ (as a function of x) in the direction of w; $D\Omega(x; u, v, w)$ is trilinear in (u, v, w). In the above expression, the $D\Omega$ term vanishes, because it contains $x - y$ in a bilinear position, and $x \sim y$. So

$$\begin{aligned} \omega(y, x, z) = \Omega(x; x - y, z - y) &= -\Omega(x; y - x, z - y) \\ &= -\Omega(x; y - x, (z - x) + (x - y)) \\ &= -\Omega(x; y - x, z - x) - \Omega(x; y - x, x - y); \end{aligned}$$

here, the last term vanishes due to the bilinear occurrence of $x - y$ and $x \sim y$, and so we may continue

$$= -\Omega(x; y - x, z - x) = -\omega(x, y, z).$$

This proves the sign change that we claimed. One notes that in the proof as presented, z (and $z - x$, $z - y$) play a "dummy" role; in fact, by replacing z by a $k-1$-tuple $z_2, \ldots, z_k$, one gets the proof for k-forms. In particular, omitting z altogether gives the proof that 1-forms are alternating, i.e. proves the validity of equation (3).

The following is essentially from [7], Section 5, but some of the assertions there were not proved, so for completeness, we include proofs here. The construction to follow is relative to some bilinear $A \times A \xrightarrow{*} A$ (we are interested in the case where $*$ is either the associative multiplication : $A \times A \xrightarrow{\cdot} A$, or is the algebraic commutator, $a * b = [a, b] = ab - ba$).

Let ω be a combinatorial A-valued k-form on M, and let θ be a combinatorial A-valued l-form on M. For any infinitesimal $k + l$-simplex
$(x_0, \ldots, x_k, x_{k+1}, \ldots, x_{k+l})$, we consider for each $j = 0, \ldots, k$ the expression

$$\omega(x_0, \ldots, x_k) * \theta(x_j, x_{k+1}, \ldots, x_{k+l}). \quad (4)$$

Lemma 7 *These expressions have same value, independently of the $j = 0, \dots, k$ chosen. Furthermore, the expression vanishes if some x_r ($r \geq k+1$) equals some x_j ($j \leq k$).*

Proof. For the first assertion: the proof is likewise a Taylor expansion argument, assuming, as we may, that M is an open subset of R^n. Like in the previous proof, we have functions Ω and Θ,

$$\Omega(- : -, \dots, -) : M \times (R^n)^k \to A,$$

$$\Theta(-; -, \dots, -) : M \times (R^n)^l \to A,$$

both being multilinear and alternating in the arguments after the semicolon, and with

$$\omega(x_0, x_1, \dots, x_k) = \Omega(x_0; x_1 - x_0, \dots, x_k - x_0),$$

and similarly for θ and Θ. Because of the symmetry properties of ω, it suffices to prove that we get the same value for $j = 0$ and $j = 1$. Let us write x for x_0 and y for x_1, and z for x_{k+1} (we don't write the remaining arguments). Then

$$\begin{aligned} \omega(x, y, \dots) * \theta(y, z, \dots) &= \Omega(x; y - x, \dots) * \Theta(y; z - y, \dots) \\ &= \Omega(x; y - x, \dots) * \Big(\Theta(x; z - y, \dots) + D\Theta(x; z - y, \dots, y - x)\Big), \end{aligned}$$

by Taylor expansion at x of Θ in its non-linear variable, in the direction of $y-x$. Here, $D\Theta$ has $l+1$ variables after the semicolon, the last (new) one being the direction in which we Taylor expand. It is multilinear in these $l+1$ variables. Multiplying out (using that $*$ is bilinear), we see that the term containing the $D\Theta$-factor vanishes, due to occurrence of $y - x$ twice (and using $y \sim x$). So the equation continues

$$\begin{aligned} &= \Omega(x; y - x, \dots) * \Theta(x; z - y, \dots) \\ &= \Omega(x; y - x, \dots) * \Big(\Theta(x; z - x, \dots) + \Theta(x; x - y, \dots)\Big); \end{aligned}$$

multiplying out (again using that $*$ is bilinear), one of the two terms contains $\pm(y-x)$ twice, and therefore vanishes, so we continue

$$= \Omega(x; y - x, \dots) * \Theta(x; z - x, \dots) = \omega(x, y, \dots) * \theta(x, z, \dots)$$

proving the first assertion of the Lemma. The second assertion is a formal consequence: if x_r for $r > k$ equals x_j for $j \leq k$, we use the "independence of j" already proved so that the product equals $\omega(x_0, \dots, x_k) \cdot \theta(x_j, x_{k+1}, \dots, x_r, \dots)$, but now the θ factor is 0, due to the repeated occurrence of $x_j = x_r$.

With ω and θ k- and l-forms, as above, we can therefore manufacture a $k + l$ form $\omega \cup_* \theta$ by the recipe

$$(\omega \cup_* \theta)(x_0, \dots, x_{k+l}) := \omega(x_0, \dots, x_k) * \theta(x_k, x_{k+1}, \dots, x_{k+l}). \tag{5}$$

For, this expression vansihes if $x_j = x_r$ for some $j, r \leq k$, because then the ω factor vanishes, and it vanishes if $x_j = x_r$ for some $r, j > k$, because then the θ factor vanishes; and it vanishes if $x_j = x_r$ with $j \leq k$ and $r > k$, by the Lemma.

Note that the formula (5) is identical to the formula that defines cup products of simplicial cochains, and many of its properties are proved the same way. It is related to the exterior product $\wedge$ of classical differential forms, and was in fact denoted by $\wedge$ in [7]. The relationship has been expounded in [9].

With $G \subseteq A$ as before, we shall prove that $(G, \cdot)$-valued 1-forms ω are likewise alternating, in the sense that

$$\omega(x, y) = (\omega(y, x))^{-1}. \tag{6}$$

For, let $l\omega$ denote the $(A, +)$-valued 1-form given by

$$(l\omega)(x, y) = \omega(x, y) - e = l(\omega(x, y))$$

(recalling the map $l : G \to A$, $g \mapsto g - e$). Then $(l\omega)(x, y) = -(l\omega)(y, x)$, by the alternating property for $(A, +)$-valued forms. So

$$\begin{aligned} \omega(x, y) = e + (l\omega(x, y)) &= e - (l\omega(y, x)) \\ &= (e + (l\omega(y, x)))^{-1} \end{aligned}$$

(by Proposition 3, using $\omega(x, y) \sim e$ in A, see the Remark below)

$$= \omega(y, x)^{-1}.$$

We define the *coboundary* $d(\omega)$ (or just $d\omega$) of a $(G, \cdot)$-valued 1-form ω on M to be the law which to an infinitesimal 2-simplex (x, y, z) in M associates $\omega(x, y) \cdot \omega(y, z) \cdot \omega(z, x)$,

$$(d\omega)(x, y, z) := \omega(x, y) \cdot \omega(y, z) \cdot \omega(z, x),$$

and it is easy to see (using (6)) that $d\omega$ is in fact a combinatorial 2-form, i.e. that it vanishes if two of its inputs are equal. In the case that the value group is $(A, +)$, the formula for $d\omega$ reads (additive notation)

$$d\omega(x, y, z) = \omega(x, y) + \omega(y, z) + \omega(z, x) = \omega(x, y) + \omega(y, z) - \omega(x, z),$$

using (3). Note that this is the usual formula for coboundary of simplicial 1-cochains, and as such immediately generalizes to $k > 1$. (For the case of non-commutative $(G, \cdot)$, a generalization to $k > 1$ is not immediately evident, but see [7] Section 11 for some use of it.)

Let us also for completeness make explicit the notion of combinatorial G-valued 0-form on a manifold M: it is just a *function* $f : M \to G$. We have its *coboundary* df, which is a combinatorial G-valued 1-form on M, given by

$$df(x, y) := f(x)^{-1} \cdot f(y).$$

It is trivial to verify that $d(d(f)) = e$, the combinatorial 2-form with constant value $e \in G$.

In the non-commutative case (which is our main interest here), the notion of coboundary of 0- and 1-forms with values in G actually come in two versions, a "left" and a "right" coboundary, d_L and d_R; we have chosen one of them, the one we call the "left", $d = d_L$. For completeness, let us write the formulas for $d = d_L$ as well as for d_R:

$$d_L f(x, y) = f(x)^{-1} \cdot f(y); \quad d_L\omega(x, y, z) = \omega(x, y) \cdot \omega(y, z) \cdot \omega(z, x),$$

$$d_R f(x, y) = f(y) \cdot f(x)^{-1}; \quad d_R\omega(x, y, z) = \omega(z, x) \cdot \omega(y, z) \cdot \omega(x, y),$$

for f a 0-form and ω a 1-form.

Remark. It does not immediately follow from the definitions that for a 1-form ω, with values in $(G, \cdot)$, say, we have $\omega(x, y) \sim e$; it *will* follow if ω – apriori defined only on the set $M_{(1)} \subseteq M \times M$ of pairs of neighbour points – extends to a map Ω on some open subset U of $M \times M$ containing $M_{(1)}$; for, any map defined on U preserves the neighbour relation, and since $(x, x) \sim (x, y)$ in $U \subseteq M \times M$, it follows that $e = \omega(x, x) = \Omega(x, x) \sim \Omega(x, y)$. Such extensions always exist if G is a finite dimensional manifold, or if it is a Euclidean vector space. So whenever it does not already follow, I shall state as a *separate* requirement, that the values of a combinatorial differential form takes values which are $\sim e$.

The argument that $\omega(x, y) = \omega(y, x)^{-1}$, however, depended only on $l\omega(x, y) \sim 0_A$, which *does* hold in any case, because A is Euclidean.

A combinatorial G-valued 1-form ω gives rise to a combinatorial A-valued 1-form $l\omega$, $l\omega(x, y) := \omega(x, y) - e$. Similarly for G-valued combinatorial 2-forms θ on M: $(l\theta)(x, y, z) := \theta(x, y, z) - e$. We shall below calculate $d(\omega)$ in terms of $d(l\omega)$. Here, both occurrences of "d" denote the combinatorial (simplicial) coboundary, but with respect to, respectively, multiplication in G and addition in A. To remind the reader of this, we write $d.$ and d_+, respectively, for these coboundaries. (In Section 8, we shall also consider the *exterior derivative* $\overline{d}$ for *classical* differential forms, and compare it with d_+.)

To make the desired comparison between the $(G, \cdot)$-valued coboundary of ω, and the $(A, +)$-valued coboundary of $l\omega$, consider an infinitesimal 2-simplex x, y, z in M. Then

$$\begin{aligned} d.\omega(x, y, z) &= \omega(x, y) \cdot \omega(y, z) \cdot \omega(z, x) \\ &= (e + l\omega(x, y)) \cdot (e + l\omega(y, z)) \cdot (e + l\omega(z, x)). \end{aligned}$$

This we may multiply out, using the distributive law in the algebra A; we get

$$\begin{aligned} &e + l\omega(x, y) + l\omega(y, z) + l\omega(z, x) \\ &\quad + l\omega(x, y) \cdot l\omega(y, z) + l\omega(x, y) \cdot l\omega(z, x) + l\omega(y, z) \cdot l\omega(z, x)) \\ &\quad + l\omega(x, y) \cdot l\omega(y, z) \cdot l\omega(z, x). \end{aligned} \tag{7}$$

Lemma 8 *The three terms in the middle line are equal except for sign. The term in the third line is 0.*

Proof. We recognize the three terms in the middle line as values of the cup product 2-form $l\omega \cup. \, l\omega$, applied to permutation instances of x, y, z – for the middle term, this may require an argument:

$$l\omega(x, y) \cdot l\omega(z, x) = (-l\omega(y, x)) \cdot (-l\omega(x, z)) = (l\omega \cup. \, l\omega)(y, x, z)$$

by cancelling the two minus signs. Now (y, x, z) is an odd permutation, the two other terms come from even permutation instances; the fact that $l\omega \cup_. l\omega$ is alternating (Proposition 6) then gives the first assertion of the Lemma. For the second assertion, we consider the A-valued 3-form $l\omega \cup_. l\omega \cup_. l\omega$ (for associative $A \times A \xrightarrow{\cdot} A$, like here, the corresponding $\cup_.$ is associative); we recognize the last line in (7) as this 3-form applied to the 3-simplex (x, y, z, x), but since this simplex has a repeated entry, the form vanishes on it. This proves the Lemma.

So in the expression (7) for $d.\omega$, only the first line and one term from the second line survive, so we have

$$\begin{aligned}(d.\omega)(x, y, z) &= e \ + \ l\omega(x, y) + l\omega(y, z) + l\omega(z, x) + l\omega(x, y) \cdot l\omega(y, z) \\ &= e + (d_+ l\omega)(x, y, z) + l\omega(x, y) \cdot l\omega(y, z),\end{aligned}$$

or

Theorem 2 *Let ω be a 1-form on a manifold M, with values in $G \subseteq A$, and let $l\omega(x, y) = \omega(x, y) - e$, and similarly for 2-forms. Then for any infinitesimal 2-simplex x, y, z in M,*

$$l(d.\omega)(x, y, z) = d_+ l\omega(x, y, z) + l\omega(x, y) \cdot l\omega(y, z). \tag{8}$$

This Theorem is the desired comparison. Note again that the $d.$ on the left hand side utilizes the multiplication $\cdot$ of G, the d_+ on the right hand side utilzes the $+$ of A.

– The second term on the right hand side can be written in several other ways; for, the three terms in the middle line of (7) are equal (modulo sign), as we proved. There is even a further doubling of the number of ways it can be written. For, we claim that for any infinitesimal 2-simplex x, y, z

$$l\omega(x, y) * l\omega(y, z) = -l\omega(y, z) * l\omega(x, y). \tag{9}$$

This follows from Proposition 4, using that $\omega(x, y)$, $\omega(z, y)$ and $\omega(y, y)$ $(= 0)$ form an infinitesimal 2-simplex in A.

The expressions for $l(d.\omega)(x, y, z)$ can be further rewritten, utilizing the cup product of forms with respect to a suitable bilinear map $A \times A \xrightarrow{*} A$,

$$(l\omega \cup_* l\theta)(x, y, z) := l\omega(x, y) * l\theta(y, z).$$

We have in mind that $*$ is the algebraic commutator $[-, -]$ on A. From (9) it follows that

$$[l\omega(x, y), l\omega(y, z)] = l\omega(x, y) \cdot l\omega(y, z) - l\omega(y, z) \cdot l\omega(x, y) = 2l\omega(x, y) \cdot l\omega(y, z),$$

whence the conclusion of the Theorem may also be written

$$ld.\omega(x, y, z) = d_+ l\omega(x, y, z) + \tfrac{1}{2}(l\omega \cup_{[-,-]} l\omega)(x, y, z);$$

so

Theorem 3 *Let ω be a 1-form on a manifold M, with values in $G \subseteq A$, and let $l\omega(x, y) = \omega(x, y) - e$, and similarly for 2-forms. Then for any infinitesimal 2-simplex x, y, z in M,*

$$ld.\omega = d_+ l\omega + \tfrac{1}{2} l\omega \cup_{[-,-]} l\omega. \tag{10}$$

6 Affine combinations

As will be expounded in [14] (see also [8] for a partial account), any map (not necessarily linear) from a finite dimensional vector space R^n (or an open subset U thereof) to a KL vector space A preserves affine combinations of mutual neighbour points $x_0, \dots, x_k$. It follows that for a finite dimensional manifold M, affine combinations of any $k+1$-tuple of mutual neighbour points make invariant sense, and are preserved by any map between such manifolds.

Here, we shall only need the case where $k = 1$, and so we may as well give a complete account (and a slight generalization) in this special case.

First we note that if $a \sim b$ in R^n, then any affine combination $(1-t)a+tb$ is $\sim a$ and $\sim b$ ($t \in R$ an arbitrary scalar). To prove the assertion about a, say, we have to prove that $((1-t)a+tb) - a \in D(n) \subseteq R^n$. Calculating this linear combination, we get $a - ta + tb - a = t(b-a)$; but $b - a$ is in $D(n)$ by the assumption $a \sim b$, and hence so is any scalar multiple of it.

Since $U \subseteq R^n$ was assumed open, we therefore also get that $(1-t)a+tb \in U$.

Now consider an arbitrary map $h : U \to A$. Consider the map $D(n) \to A$ given by $\underline{d} \mapsto h(a + \underline{d})$. By the KL axiom for A, it is of the form $d \mapsto h(a) + L(\underline{d})$ for a unique linear $L : R^n \to A$. In particular, the restriction of h to the subset $a + D(n) \subseteq U \subseteq R^n$ extends (uniquely) to an affine map $H : R^n \to A$. Since b and also $(1-t)a+tb$ belong to the subset $a + D(n)$ (being neighbours of a), the values of h and H agree on these three points. But H preserves affine combinations. So $h : U \to A$ preserves affine combinations of pairs of neighbour points.

Proposition 9 *Let $i : G \to A$ be a monic map into a Euclidean vector space. Assume $x \sim y \in G$. Then the affine combination*

$$(1-t) \cdot i(x) + t \cdot i(y)$$

is of the form $i(z)$ for some $z \in G$ (necessarily unique). This z satisfies $z \sim x$ and $z \sim y$; and it is independent of the embedding i, i.e. any other injective $i' : G \to A'$ into a Euclidean vector space A' will produce the same $z \in G$. We write this z as $(1-t) \cdot x + t \cdot y$.

Proof. By the assumption that $x \sim y \in G$, there is some open $U \subseteq R^n$, some $a \sim b \in U$ and some $f : U \to G$ with $f(a) = x$ and $f(b) = y$. We saw above that $(1-t)a+tb$ (formed in R^n) is $\sim a$ and hence belongs to U. Let $z := f((1-t)a+tb))$. Then

$$i(z) = i(f((1-t) \cdot a + t \cdot b)) = (1-t) \cdot i(f(a)) + t \cdot i(f(b))$$

since $i \circ f : U \to A$ preserves affine combinations of mutual neighbour points (as we saw above for an arbitrary $h : U \to A$); so $i(z) = (1-t) \cdot i(x) + t \cdot i(y)$, proving the existence of z. The fact that $z \sim x$ follows because f preserves the neighbour relation, in particular the relation $(1-t)\cdot a + t \cdot b \sim a$. Similarly for $z \sim y$. Finally for the independence assertion: the construction $z := f((1-t) \cdot a + t \cdot b)$ is independent of i, and the description of z via $i(z) = (1-t) \cdot i(x) + t \cdot i(y)$ is independent of f, so z is independent of choice of both f and i.

Using affine combinations, we now have a process which to a pair (x, y) of neighbour points in a manifold M associates a tangent vector at x, denoted $\log(x, y)$; it is given by

$$\log(x, y)(d) := (1 - d) \cdot x + d \cdot y.$$

7 From G-valued forms to $\mathfrak{g}$-valued forms

We shall consider combinatorial differential forms ω with values in $G \subseteq A$; as noted in the Remark in Section 5, their values are $\sim e$ when seen in A, but not necessarily when seen in G. This we have to assume. It is a very weak assumption that can be seen to hold under various kinds of assumptions, e.g. assuming that G is a finite dimensional manifold.

Let us denote the set of neighbours of $e \in G$ by $\mathcal{M}_1(e)$. For $g \in \mathcal{M}_1(e)$, we have an element $\lambda(g) \in T_e(G) = \mathfrak{g}$, namely $\lambda(g) := \log(e, g)$, so

$$\lambda(g)(d) = \log(e, g)(d) = (1 - d) \cdot e + d \cdot g;$$

this affine combination is in G, by Proposition 9, which also gives that $\lambda(g)$ is independent of the auxiliary embedding $G \subseteq A$. Clearly $\lambda(e)$ is the zero vector at e, i.e. the map $d \mapsto e \; \forall d \in D$.

Recall the map $L : \mathfrak{g} \to A$ which associates to $\tau \in \mathfrak{g} = T_e(G)$ the principal part of $d \mapsto \tau(d) - e$; and recall the map $l : G \to A$ with $l(g) = g - e$. We have

$$L \circ \lambda = l. \tag{11}$$

For, let $g \sim e$ in G. Then $L(\lambda(g))$ is the principal part of the tangent vector of A given by $d \mapsto (1 - d) \cdot e + d \cdot g = e + d \cdot (g - e)$, and this principal part is clearly $g - e$, i.e. $l(g)$.

Since the values of G-valued k-forms θ are in $\mathcal{M}_1(e)$, it follows that we may compose such θ with λ to obtain a $\mathfrak{g}$-valued combinatorial k-form $\lambda\theta$.

Let $l\omega$ and $l\theta$ be A-valued combinatorial 1-forms. Recall that one defines an A-valued combinatorial 2-form $l\omega \cup_{[-,-]} l\theta$ by putting

$$l\omega \cup_{[-,-]} l\theta := [l\omega(x, y), l\theta(y, z)] \tag{12}$$

where the square brackets denote algebraic commutator in A, $[u, v] = uv - vu$. Similarly if $\lambda\omega$ and $\lambda\theta$ are $\mathfrak{g}$-valued 1-forms, we have a $\mathfrak{g}$-valued 2-form $\lambda\omega \cup_{[[-,-]]} \lambda\theta$, like in (12) with $[[-, -]]$ instead of $[-, -]$, and $\lambda\omega$, $\lambda\theta$ instead of $l\omega$, $l\theta$.

Now let ω be a G-valued 1-form. So we have $\lambda\omega$, a $\mathfrak{g}$-valued 1-form, and we have $l\omega$, an A-valued 1-form; and the former goes to the latter by $L : \mathfrak{g} \to A$, because of (11).

We can therefore get a Corollary of Theorem 3, namely

Theorem 4 *Let ω be a G-valued 1-form. Then we have the following equality of combinatorial $\mathfrak{g}$-valued 2-forms:*

$$\lambda d.\omega = d_+\lambda\omega + \tfrac{1}{2}\lambda\omega \cup_{[[-,-]]} \lambda\omega.$$

Here $G \subset A$, as always, but note that the entries in the equation of the Theorem no longer depend on A, but are intrinsic to G (and to $\mathfrak{g}$, which in turn is intrinsic to G).

8 Comparison between classical and combinatorial forms

In this section, we consider differential forms with values in a vector space A, assumed to be Euclidean.

The forms we consider are defined on a finite dimensional manifold M. Therefore, we have, for any pair of neighbour points x_0, x_1 in M a well defined tangent with base point x_0, $\log(x_0, x_1)$, as in the Section 6: $\log(x_0, x_1)(d) := (1-d)x_0 + dx_1$ for $d \in D$.

Recall that a classical differential k-form $\overline{\omega}$ on M (with values in a Eucldean vector space A) is a law which to a k-tuple of tangents, say $\tau_1, \ldots, \tau_k$, with common base point, associates an element $\overline{\omega}(\tau_1, \ldots, \tau_k) \in A$ in a k-linear and alternating way.

We shall recall a bijective correspondence between classical and combinatorial k-forms (cf. [4] I.18 for this correspondence in coordinates, and [13] for the correspondence in coordinate free terms); the combinatorial k-form ω corresponding to the classical k-form $\overline{\omega}$ is given by

$$\omega(x_0, x_1, \ldots, x_k) := \overline{\omega}(\log(x_0, x_1), \ldots, \log(x_0, x_k))$$

for $(x_0, \ldots, x_k)$ an infinitesimal k-simplex.

To see that this correspondence is bijective, it suffices to assume that M is in fact an open subset of $V = R^n$, for some n. A classical A-valued k-form $\overline{\omega}$ may then be encoded into a function $F : M \times V^k \to A$, which is k-linear and alternating in the last k arguments:

$$F(x_0; v_1, \ldots, v_k) = \overline{\omega}(\tau_1, \ldots, \tau_k),$$

where τ_is are tangent vectors with base point $x_0 \in M \subseteq V$ and with principal part $v_i \in V$ (so $\tau_i(d) = x_0 + d \cdot v_i$).

On the other hand, a combinatorial A-valued k-form ω gives rise function $f : M \times \widetilde{D}(k, V) \to A$, via

$$\omega(x_0, x_1, \ldots, x_k) = f(x_0; x_1 - x_0, \ldots, x_k - x_0) \tag{13}$$

with f vanishing if one of the k arguments after the semicolon vanishes. But a function $f(x_0; -, \ldots, -) : \widetilde{D}(k, V) \to A$ with this property extends uniquely to a k-linear alternating function $F(x_0; -, \ldots, -) : V^k \to A$. Collectively, these $F(x_0; -, \ldots, -)$ define a function $F(-; -, \ldots, -) : M \times V^k \to A$, k-linear and alternating in the arguments after the semicolon. (This F is what in Section 5 was denoted Ω.)

Remark. Note that the assumptions on ω needed to produce such F was just the "weak" one: that $\omega(x_0, \ldots, x_k) = 0$ if some x_i with $i > 0$ equals x_0. But from the fact that F is alternating in the last k arguments then *follows* the stronger conclusion: the vanishing of $\omega(x_0, x_1, \ldots, x_k)$ if some $x_i = x_j$ with $i, j \geq 1$. The reason why we take the stronger condition in our definition of the notion of combinatorial differential form is just that it is simpler to state, and is more symmetrical.

We see that both $\overline{\omega}$ and ω are encoded by the function $F(-;-,\ldots,-) : M \times V^k \to A$, k-linear and alternating in the arguments after the semicolon.

This already establishes a bijective correspondence between classical and combinatorial k-forms, but does not prove that the correspondence described via such F is "independent of choice of coordinates". To see this independence, we just have to argue that the correspondence that we have set up in terms of functions $F(-;-,\ldots,-) : M \times V^k \to A$ agrees with the one that we have set up in terms of the tangent vectors $\log(x_0, x_i)$. This is an easy consequence of the fact that the principal part of the tangent vector $\log(x_0, x_i)$ is $x_i - x_0$ (see the argument for (11)). So if $\overline{\omega}$ is a classical k-form (encoded by the function F), and ω the combinatorial k-form corresponding to it by the "log"-correspondence,

$$\begin{aligned}\omega(x_0, x_1, \ldots, x_k) &= \overline{\omega}(\log(x_0, x_1), \ldots, \log(x_0, x_k)) \\ &= F(x_0; x_1 - x_0, \ldots, x_k - x_0),\end{aligned}$$

which is the combinatorial k-form encoded by F.

We next want to compare the coboundary $d_+(\omega)$ of vector space valued combinatorial forms (which was defined by the standard "simplicial" formula) with the exterior derivative $\overline{d}(\overline{\omega})$ of classical differential forms. We now write $d(\omega)$ for $d_+(\omega)$, for simplicity. So let A be a Euclidean vector space. We have

Theorem 5 *Let ω be an A-valued combinatorial k-form on a manifold M, and let $\overline{\omega}$ be the corresponding classical form. Then the combinatorial $k+1$-form $d(\omega)$ has $1/(k+1)\ \overline{d}(\overline{\omega})$ for its corresponding classical $k+1$-form.*

(In [9], the factor $k+1$ does not appear; this is because the correspondence between combinatorial and classical forms is set up differently there, with a scalar factor built in.)

Proof. For a proof for general k, see [13]; we shall only prove the case $k = 1$, which is the relevant one here. It suffices to do it in the coordinatized situation, i.e. with M an open subset of $V = R^n$, so both ω and $\overline{\omega}$ are encoded by a function $F : M \times V \to A$, linear in the second argument. We first calculate $d\omega(x, y, z)$ for an infinitesimal 2-simplex (x, y, z) in M:

$$\begin{aligned}d\omega(x, y, z) &= \omega(x, y) + \omega(y, z) - \omega(x, z) \\ &= F(x; y - x) + F(y; z - y) - F(x; z - x).\end{aligned}$$

We now Taylor expand the middle term (as a function in the variable in front of the semicolon) from x in the direction of $y - x$; this term then is $F(x; z - y) + DF(x; z-y, y-x)$, where $DF(-;-,v)$ denotes directional derivative in the direction of $v \in V$; it depends in a linear way on v, which is why we put the variable v after the semicolon. With this expansion, we get

$$d\omega(x, y, z) = F(x; y - x) + F(x; z - y) - F(x; z - x) + DF(x; z - y, y - x).$$

Using linearity of F in the argument after the semicolon, one sees that the three first terms cancel out; so only the DF term remains; we proceed to calculate this term, (writing $DF(u,v)$ for $DF(x;u,v)$). We have

$$\begin{aligned} DF(z-y, y-x) &= DF((z-x)+(x-y), y-x) \\ &= DF(z-x, y-x) + DF(x-y, y-x). \end{aligned}$$

The last term here vanishes because DF is bilinear, and $y \sim x$. So we are left with the expression for $d\omega$ in terms of F:

$$d\omega(x,y,z) = DF(x; z-x, y-x,). \tag{14}$$

We can get an alternative expression, utilizing that $y \sim z$ and hence $(y-x) \sim (z-x)$ (and both ~ 0); for it then follows that the bilinear $DF(x; -,-)$ behaves as if it were alternating when applied to $(y-x), (z-x)$, so $DF(x; z-x, y-x) = -DF(x; y-x, z-x)$. So

$$d\omega(x,y,z) = \tfrac{1}{2}\big(DF(x; z-x, y-x) - DF(x; y-x, z-x)\big). \tag{15}$$

Now, the function $DF(x;u,v) - DF(x;v,u)$ is clearly bilinear alternating in u, v, and as such represents a classical 2-form on M; and this 2-form is the (coordinate expression for) the classical exterior derivative $\overline{d}$ of the 1-form $\overline{\omega}$ given by F (see e.g. [16] p. 15). This proves the Theorem (for the case $k=1$).

For the case where the value vector space A for the (combinatorial, as well as classical) forms is equipped with a bilinear $* : A \times A \to A$, we would like to compare the $\cup_*$-product of combinatorial forms (introduced above in Section 5) with the classical wedge product (relative to $*$) of classical forms, which we denote $\wedge_*$. Recall that if $\overline{\omega}$ and $\overline{\theta}$ are, respectively a classical k-form and a classical l-form (on M, with values in A), then we get a classical $k+l$-form $\overline{\omega} \wedge_* \overline{\theta}$ given by

$$\begin{aligned} &(\overline{\omega} \wedge_* \overline{\theta})(\tau_1, \ldots, \tau_{k+l}) := \\ &\qquad \sum_{\sigma \in S(k,l)} \operatorname{sign}(\sigma)\, \overline{\omega}(\tau_{\sigma(1)}, \ldots, \tau_{\sigma(k)}) * \overline{\theta}((\tau_{\sigma(k+1)}, \ldots, \tau_{\sigma(k+l)}), \end{aligned}$$

where $S(k,l)$ denotes the set consisting of the $(k+l)!/\ k!\ l!$ "(k,l)-shuffles" (permutations of $\{1, \ldots, k+l\}$, which keep the mutual order of $1, \ldots, k$ and also keep the mutual order of $k+1, \ldots, k+l$). If the τ_i are of the form $\log(x_0, x_i)$ for an infinitesimal $k+l$ simplex $x_0, x_1, \ldots, x_{k+l}$, then the sum may be rewritten, term by term, using the combinatorial form ω corresponding to $\overline{\omega}$, as

$$\sum_{\sigma \in S(k,l)} \operatorname{sign}(\sigma)\, \omega(x_0, x_{\sigma(1)}, \ldots, x_{\sigma(k)}) * \overline{\theta}(x_0, x_{\sigma(k+1)}, \ldots, x_{\sigma(k+l)}).$$

We recognize the individual terms here as instances of $\omega \cup_* \theta$; because $\omega \cup_* \theta$ is alternating, all the terms in this sum are equal, so the sum equals

$$(k+l)!/\ k!\ l!\ (\omega \cup_* \theta)(x_0, \ldots, x_{k+l}).$$

We shall only need this for $k = l = 1$; there are two $(1,1)$-shuffles, so if ω and θ are combinatorial 1-forms, and $\overline{\omega}, \overline{\theta}$ the corresponding classical forms, we therefore have that

$$\overline{\omega} \wedge_* \overline{\theta} \quad \text{corresponds to} \quad 2(\omega \cup_* \theta). \tag{16}$$

Note that if $*$ is skew, $u * v = -v * u$, then we have the second equality sign in

$$(\overline{\omega} \wedge_* \overline{\theta})(\tau_1, \tau_2) = \overline{\omega}(\tau_1) * \overline{\theta}(\tau_2) - \overline{\omega}(\tau_2) * \overline{\theta}(\tau_1) = \overline{\omega}(\tau_1) * \overline{\theta}(\tau_2) + \overline{\theta}(\tau_1) * \overline{\omega}(\tau_2),$$

and in particular, for $\overline{\omega} = \overline{\theta}$,

$$(\overline{\omega} \wedge_* \overline{\omega})(\tau_1, \tau_2) = 2\omega(\tau_1) * \omega(\tau_2).$$

9 Group valued coboundary vs. Lie algebra valued exterior derivative

The ultimate comparison we want is between combinatorial G-valued 1- and 2-forms θ, on the one hand, and the corresponding classical $\mathfrak{g}$-valued 1- and 2-forms $\overline{\lambda\theta}$, on the other. The correspondence itself is in two stages: first, the process λ, which takes combinatorial G-valued forms to combinatorial $\mathfrak{g}$-valued forms, and secondly the the process denoted by "overline", which takes combinatorial forms to classical forms (and which applies to vector space valued forms only, so cannot be applied directly to G-valued forms).

More precisely, given a combinatorial G-valued 1-form ω, we want a comparison between its coboundary $d(\omega)$, which is a combinatorial G-valued 2-form, and the classical $\mathfrak{g}$-valued 2-form $\overline{d}(\overline{\lambda\omega})$, $\overline{d}$ denoting the classical exterior derivative. It results in a formula, which is identical to (5.2) from [5], but now proved in bigger generality and in coordinate free terms.

The work has essentially been carried out above, by Theorem 4 and Theorem 5. For the combinatorial coboundaries, we shall again use the notation $d.$ and d_+ to remind ourselves whether we are talking about G-valued, or vector valued forms.

Theorem 6 *Let ω be a combinatorial G-valued 1-form. Then we have the following equality of classical $\mathfrak{g}$-valued 2-forms:*

$$\overline{\lambda(d.\omega)} = \tfrac{1}{2}\big(\overline{d}\;\overline{\lambda\omega} + \tfrac{1}{2}\overline{\lambda\omega} \wedge_{[[-,-]]} \overline{\lambda\omega}\big).$$

In simplified notation, let $\wedge$ denote $\wedge_{[[-,-]]}$, and let $\widetilde{\theta}$ denote the combined construction $\theta \mapsto \overline{\lambda\theta}$. Then the equation can also be written

$$\widetilde{d.(\omega)} = \tfrac{1}{2}\big(\overline{d}\widetilde{\omega} + \tfrac{1}{2}\widetilde{\omega} \wedge \widetilde{\omega}\big).$$

We shall partially use the simplified notation also in the proof to be given. Then $\cup$ denotes $\cup_{[[-,-]]}$.

Proof. By Theorem 4

$$\lambda(d.\omega) = d_+\lambda\omega + \tfrac{1}{2}\lambda\omega \cup \lambda\omega.$$

Since the "overline"-process is clearly linear, we therefore have

$$\overline{\lambda(d.\omega)} = \overline{d_+\lambda\omega} + \tfrac{1}{2}\ \overline{\lambda\omega \cup \lambda\omega}$$
$$= \tfrac{1}{2}\overline{d}(\overline{\lambda\omega}) + \tfrac{1}{2}\ \overline{\lambda\omega \cup \lambda\omega}$$

(by Theorem 5)

$$= \tfrac{1}{2}\overline{d}(\overline{\lambda\omega}) + \tfrac{1}{2}\ (\tfrac{1}{2}\ \overline{\lambda\omega} \wedge \overline{\lambda\omega})$$

(by (16))

$$= \tfrac{1}{2}\ \Big(\overline{d}(\overline{\lambda\omega}) + \tfrac{1}{2}\ \overline{\lambda\omega} \wedge \overline{\lambda\omega}\Big).$$

This proves the Theorem.

Example. ([5], Coroll. 5.5.) Let G be a Lie group, in particular, it is a finite dimensional manifold. Then there is a canonical G-valued 0-form on G, namely the identity map $i : G \to G$. Its coboundary ω is the G-valued 1-form on G given by $\omega(x, y) = x^{-1} \cdot y$. The classical $\mathfrak{g}$-valued 1-form on G corresponding to this ω is the the Maurer-Cartan form, denoted Ω, thus

$$\Omega := \overline{\lambda\omega} = \widetilde{\omega}.$$

Now $\omega = d.(i)$ is trivially closed, in the sense that $d.(\omega) = e$, (= the 2-form with constant value e). The corresponding classical 2-form is therefore the zero 2-form. The formula of Theorem 6 therefore has 0 on its left hand side, and so the right hand side is 0 as well, and therefore

$$\overline{d}\Omega = -\tfrac{1}{2}\Omega \wedge \Omega,$$

the classical Maurer-Cartan formula.

10 Curve integrals of 1-forms

Having the process which to a function $f : M \to G$ associates a (closed) G-valued 1-form $d.f$ on M, one may raise the question: which closed 1-forms ω on M (or on suitable subsets of M) have "primitives" or "integrals", i.e. functions $f : M \to G$ with $d.f = \omega$? Many existence problems and results in differential geometry can be formulated in these terms, in particular for the case when M is 1-dimensional, in which case one is dealing with *curve* integrals (=path integrals). However, the generality for this question, as stated here, is not really the best one. The notion of combinatorial group-valued 1-forms should, in so far as the integration problem is concerned, better be formulated in terms of *groupoid*-valued 1-form. If $\Phi \rightrightarrows M$ is a groupoid whose object set M is a manifold, a Φ-valued 1-form ω on M is a law which to any pair of neighbour points (x, y) in M associates an arrow $\omega(x, y) : x \to y$ in Φ; the sole requirement is that $\omega(x, x)$ is the identity arrow at x. A Φ-valued 1-form (on M) is tantamount to a connection ∇ in Φ. (We recover the more special notion of G-valued 1-form on M by taking the groupoid to be the product of the trivial (chaotic) groupoid $M \times M \rightrightarrows M$ with the group G (viewed as a one-object groupoid.))

Questions of "primitives" of G-valued 1-forms generalize then to questions of *holonomy* of connections in a groupoid. This is elaborated on in [11].

11 Appendix

We remind the reader that Synthetic Differential Geometry is a reasoning that takes place in some category, where "everything is smooth", and where there is singled out a commutative ring R, to be thought of as the "number line". Typically, such a category (model of SDG) is a topos. There are several texts introducing the subject, e.g. [4] or [15].

11.1 The KL axiom, and Euclidean vector spaces

A main specific aspect is the use of an assumption of sufficiently many nilpotent elements in the ring R, in particular elements $d \in R$ with $d^2 = 0$. The set $D \subseteq R$ of these elements is required to be sufficiently big in the precise sense that the so-called "Kock-Lawvere" axiom (KL) is assumed to hold. This axiom may be formulated for an arbitrary R-module ("vector space") A, and says

> for any map $f : D \to A$, there is a unique $a \in A$ (namely $f(0)$) and a unique $b \in A$ such that
>
> $$f(d) = a + d \cdot b \quad \forall d \in D.$$

The main basic assumption in SDG is that the R-module $A = R$ itself satisfies this.

From the uniqueness of b in the axiom follows that if $d \cdot b_1 = d \cdot b_2$ for all $d \in D$, then $b_1 = b_2$. We express sometimes this by saying that "*universally quantified ds may be cancelled*".

There is also a more comprehensive KL axiom, which is really an axiom *scheme*, namely there is one instance of the Axiom for each "infinitesimal object"; see [4], I.16, where it is called Axiom 1^W, or [15] 2.1, where it is called "the general Kock axiom"; we don't need it in its full generality in the present note, but only the instances of it which refer to the infinitesimal objects $\widetilde{D}(k, n)$ which we describe below.

The comprehensive KL axiom may be formulated for an arbitrary R-modules A. If a vector space (= R-module) satisfies the comprehensive KL axiom, we call if (following [15]) a *Euclidean* vector space. The main assumption in SDG is that R itself satisfies the axiom scheme, i.e. is a Euclidean vector space. Then many other vector spaces are Euclidean as well.

Let us first be explicit about the KL axiom for $D(n)$. For a vector space A, it says

> for any map $f : D(n) \to A$ there is a unique $a \in A$ (namely $f(0)$) and a unique linear $B : R^n \to A$ such that
>
> $$f(\underline{d}) = a + B(\underline{d}) \quad \forall \underline{d} \in D(n).$$

Consider the set $\widetilde{D}(k, n) \subseteq R^{k \cdot n}$ (= the set of $k \times n$-matrices with entries from R) consisting of matrices $[x_{ij}]$ which satisfy the equations

$$x_{ij}x_{i'j'} + x_{ij'}x_{i'j} = 0 \quad \forall i, i' = 1, \ldots, k, \quad \forall j, j' = 1, \ldots, n.$$

For $k = 2$, it is the subset $\widetilde{D}(2, n)$ described in Section 2. For $k = 1$, $\widetilde{D}(1, n)$ may be identified with the set $D(n) \subseteq R^n$ likewise described in Section 2. For $k = n = 1$, we get the set $D \subseteq R$.

(The sets $\widetilde{D}(k, n)$ were introduced in [4] I.16; a deeper study of some of their algebraic properties may be found in [12].)

The comprehensive KL axiom, when applied to the object $\widetilde{D}(k, n)$, and to an R-module A says:

> for any map $f : \widetilde{D}(k, n) \to A$, there are unique elements $b_S \in A$, such that
>
> $$f(X) = \sum_S \det(X_S) \cdot b_S \quad \forall X \in \widetilde{D}(k, n)$$

here, S ranges over the set of (indices for) square submatrices (of any size) of the matrix $X \in \widetilde{D}(k, n)$, and X_S is the submatrix picked out by these indices. (The determinant of the 0×0 submatrix is taken to be 1).

Thus, for instance, if A satisfies the comprehensive KL axiom, any map $F : \widetilde{D}(2, 2) \to A$ is of the form

$$[x_{ij}] \mapsto a_0 \; + \sum_{ij} x_{ij} \cdot a_{ij} \; + \det([x_{ij}]) \cdot a,$$

for a unique six-tuple $a_0, a_{11}, a_{12}, a_{21}, a_{22}, a$ of elements of A. If the a_0 and a_{ij} are 0, $F(X)$ thus equal a times the determinant of X, thus is bilinear and alternating.

The crux in proving validity of these axioms for R in the topos models consists in providing a linear basis for the commutative ring containing the "generic" matrix of the given kind. Thus, the axiom for $\widetilde{D}(2, 2)$ is proved by proving that the ordinary commutative ring

$$\mathbb{Q}[X_{11}, X_{12}, X_{21}, X_{22}]/I$$

(where I is the ideal generated by the polynomials $X_{ij}X_{i'j'} + X_{ij'}X_{i'j}$) is finite dimensional as a vector space over $\mathbb{Q}$, with a basis consisting of the classes modulo I of the six polynomials $1, X_{11}, X_{12}, X_{21}, X_{22}, X_{11}X_{22} - X_{12}X_{21}$, the last one being the determinant of the indeterminates.

11.2 Open subsets, and manifolds

We say that a subset $U \subseteq R^n$ is *open* if it has the property that $x \in U$ and $y \sim x$ imply $y \in U$. It is related to what we called "etale subset" in [4], but "open" brings along with it the correct connotations from ordinary differential geometry, without being technical.

We do not want to be precise about how a notion of (finite dimensional) manifold could be formulated in the present context – it is again somewhat technical, and perhaps not the right notion anyway, see [4] I.17. Certainly, R^n and open subsets U thereof should be included among manifolds. Anyway, all our notions and reasoning are of "local" (even infinitesimal) nature, so is really just the coordinatefree reasoning about such open Us.

11.3 Taylor expansion

Let $U \subseteq R^n$ be open, and let A be a Euclidean vector space. If $f : U \to A$ is an arbitrary map, $x \in U$ and $u \in R^n$, we consider the map $g : D(n) \to A$ given by $\underline{d} \mapsto f(x + \underline{d}) - f(x)$. Since $g(0) = 0$, the KL axiom for A implies that g extends uniquely to a linear map $G : R^n \to A$, denoted $Df(x; -)$, thus, for $y \sim x$, $f(y) - f(x) = Df(x; y - x)$, or

$$f(y) = f(x) + Df(x; y - x). \tag{17}$$

This is the *Taylor expansion* of f from x in the direction of $y-x$. Note that $Df(-;-)$ is linear in the variable after the semicolon.

More generally, if $g : U \times (R^n)^k \to A$ is k-linear in the last k variables, then there is a function $Dg : U \times (R^n)^k \times R^n \to A$ such that, for any $y \sim x$, we have

$$g(y; v_1, \ldots, v_k) = g(x; v_1, \ldots, v_k) + Dg(x; v_1, \ldots, v_k, y - x)$$

with Dg linear in the $k + 1$ variables after the semicolon. If f is furthermore alternating in the k variables after the semicolon, then Df is also alternating w.r.to these k variables.

These assertions, one gets from the simple Taylor expansion (17), by reinterpreting g as a function

$$f = \hat{g} : U \to \mathrm{Multilin}((R^n)^k, A),$$

the codomain here being the vector space of k-linear maps $(R^n)^k \to A$, which is Euclidean if A is. (For the "furthermore" assertion, replace the codomain by $\mathrm{Multilin\text{-}alt}((R^n)^k, A)$.)

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