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A Godement theorem for locales

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The classical Godement Theorem for manifolds, characterizing kernel pairs for submersions (cf. e.g. [**15**], LG IV §5), has been used by Pradines[**12**] as a crucial property for having a good theory of differentiable groupoids; in fact, he developed an axiomatic theory of categories in which Godement's and some other exactness properties hold, under the name of 'Godement diptych'.

Establishing such theorems for locales therefore has as a corollary that the groupoid-theoretic technique developed by Pradines and others [**1**, **13**, **14**], for studying the category of 'orbital varieties' (e.g. quotient 'spaces' for foliations) becomes available for studying the category of toposes. This is because the category of orbital varieties appears as a fraction category (see [**14**]) of the category of differentiable groupoids in the same way as the category of toposes appears as a fraction category of localic groupoids (see [**9**, **10**], extending [**6**]). So, also, the notion of orbital variety may be considered as the smooth analogue of the notion of topos:

$$\frac{\text{differentiable groupoids}}{\text{orbital varieties}} = \frac{\text{localic groupoids}}{\text{toposes}}.$$

To state the results, let us fix some terminology.

By an *α-open* equivalence relation on a locale X, we mean a sublocale $i: R \hookrightarrow X \times X$ which is an equivalence relation in the standard sense (a finite limit notion), and such that the composite α of i with $\mathrm{proj}_0: X \times X \to X$ is an open surjection. We say that the equivalence relation is *closed*, *open*, etc. if $i: R \hookrightarrow X \times X$ is.

Our main result, which is a kind of Godement theorem, may be stated as follows.

THEOREM A. *Any closed α-open equivalence relation R on X is the kernel pair of its quotient map $X \to X/R$.*

This also holds when 'closed' is replaced by 'weakly closed', in a sense recently introduced by Johnstone[**5**].

Combining this with the closed-subgroup theorem of [**3**], we obtain as a corollary that *every normal subgroup of a localic group is a kernel.*

All our results, except possibly this corollary, hold in any topos; and in fact, the constructive version of the closed-subgroup theorem proved in [**5**] combines with our result to yield a constructive version of the corollary on normal subgroups.

For completeness, we shall prove also a result which is reminiscent of Theorem A, and is due to Moerdijk[**9**]:

THEOREM B. *Assume $X \to 1$ is an open surjection. Then any open (hence α-open) equivalence relation R on X is the kernel pair of its quotient map.*

Such equivalence relations are somewhat more special than those of Theorem A, since the quotient locale X/R is discrete in this case.

We use the locale/frame terminology of [**4**], but the notation of [**6**], in which the frame corresponding to a locale X is denoted $O(X)$, and the frame homomorphism corresponding to a locale map $f: X \to Y$ is denoted $f^-: O(Y) \to O(X)$. If f^- has a left adjoint, in particular if f is open, the left adjoint is denoted f^+ (where [**6**] has $\exists_f$).

We tacitly work with locales/frames in an arbitrary topos, where the initial frame is denoted Ω. For the reader who prefers to think classically (in the category of sets), Ω consists of just the two elements 0 and 1, and $\Omega \to O(X)$ sends 0 to 0 and 1 to 1.

A preliminary, somewhat different version of the present work appeared in preprint form as [**7**].

1. *Some generalities concerning α-open localic categories*

We consider a localic category, i.e. a category object in the category of locales,

$$R \underset{\beta}{\overset{\alpha}{\rightrightarrows}} X, \tag{1·1}$$

X and R being the locale of objects, respectively of arrows, and α and β being domain- and codomain-formation, respectively. We say that (1·1) is an *α-open* localic category if α is an open locale map (and so necessarily an open surjection).

We have the pull-back diagram which defines the object H of composable pairs

$$\begin{tikzcd} H \arrow[r, "a"] \arrow[d, "b"'] & R \arrow[d, "\beta"] \\ R \arrow[r, "\alpha"'] & X, \end{tikzcd} \tag{1·2}$$

and since α is an open surjection, then so is a, and the Beck–Chevalley condition

$$\beta^-\alpha^+ = a^+b^- \tag{1·3}$$

holds, cf. [**6**]. Furthermore we have the composition map $c: H \to R$ which satisfies $\alpha c = \alpha a$ and $\beta c = \beta b$, thus

$$c^-\alpha^- = a^-\alpha^- \tag{1·4}$$

$$c^-\beta^- = b^-\beta^-. \tag{1·5}$$

We also have the identity formation map $i: X \to R$ which is right inverse for both α and β, so

$$i^-\beta^- = \mathrm{id}_{O(X)} = i^-\alpha^-. \tag{1·6}$$

The subset $O(Q) \hookrightarrow O(X)$ given by

$$O(Q) = \{u \in O(X) \mid \beta^-(u) \leqslant \alpha^-(u)\} \tag{1·7}$$

is evidently a subframe, so we get a locale map $q: X \to Q$, with q^- the inclusion $O(Q) \hookrightarrow O(X)$. It clearly satisfies

$$q\beta \leqslant q\alpha,$$

and is in fact universal with this property, so it is the 'sub-coequalizer' of β and α. This does not depend on α being open, but if it is, we can say more.

Recall that a closure operator $\downarrow$ on a poset O is an order-preserving map $\downarrow: O \to O$ with

$$u \leqslant \downarrow u \quad \text{and} \quad \downarrow\downarrow u \leqslant \downarrow u \tag{1·8}$$

for all $u \in O$ (the second inequality is then in fact an equality). An element $u \in O$ is *closed* for $\downarrow$ if $u = \downarrow u$; and if $O' \hookrightarrow O$ is the set of closed elements, the inclusion q^-: $O' \hookrightarrow O$ has a left adjoint q^+ determined by $q^- q^+ = \downarrow$.

PROPOSITION 1·1. *Let* (1·1) *be an α-open localic category. Then $\alpha^+\beta^-: O(X) \to O(X)$ is a closure operator $\downarrow$, and its closed elements are the elements of $O(Q)$, described in* (1·7). *In particular, the inclusion $q^-: O(Q) \hookrightarrow O(X)$ has left adjoint q^+ which satisfies*

$$q^+q^- = \alpha^+\beta^- = \downarrow.$$

The locale map $q: X \to Q$ thus defined is characterized as the sub-coequalizer of β, α.

Proof. Let us verify the two inequalities in (1·8) for $\downarrow = \alpha^+\beta^-$. We have, for $u \in O(X)$,

$$u = i^-\beta^-(u) \leqslant i^-\alpha^-\alpha^+\beta^-(u) = \alpha^+\beta^-(u),$$

using (1·6), a front adjunction, and (1·6) again; thus $u \leqslant \downarrow u$. For the other inequality,

$$\begin{aligned} \alpha^+\beta^-\alpha^+\beta^- &= \alpha^+a^+b^-\beta^- && \text{by (1·3)} \\ &= \alpha^+a^+c^-\beta^- && \text{by (1·5)} \\ &\leqslant \alpha^+a^+c^-\alpha^-\alpha^+\beta^- \\ &= \alpha^+a^+a^-\alpha^-\alpha^+\beta^- && \text{by (1·4)} \\ &\leqslant \alpha^+\beta^-, \end{aligned}$$

where the two inequality signs are obtained by a front adjunction and two back adjunctions, respectively.

We can now easily identify the set of closed elements for $\downarrow$ as $O(Q)$, for we have $\downarrow u = u$ if and only if $\downarrow u \leqslant u$ if and only if $\alpha^+\beta^-(u) \leqslant u$ if and only if $\beta^-(u) \leqslant \alpha^-(u)$ (by adjunction $\alpha^+ \dashv \alpha^-$).

The conclusion of the proposition then follows, by the above generalities about closure operators.

If (1·1) is an α-open localic groupoid, the codomain map β is open as well, using the inversion map $R \to R$. By symmetry, it follows from Proposition 1·1 that $\beta^+\alpha^-$ is also a closure operator $\uparrow$ on $O(X)$, but we also have

LEMMA 1·2. *For an α-open localic groupoid, the closure operators $\alpha^+\beta^-$ and $\beta^+\alpha^-$ on $O(X)$ agree.*

Proof. For a groupoid object in a category with finite limits, we have many more pull-backs than (1·2); we shall exploit three of these (which occur in [**2**]). We identify a localic groupoid with its nerve, which is a simplicial locale G_*.

Thus, for the groupoid (1·1), $G_0 = X$, $G_1 = R$, and G_2 is the object H of composable pairs that occurs in (1·2); the face operator $d_i: G_{n+1} \to G_n$ 'projects away' the ith factor, so $\alpha = d_1$, $\beta = d_0$; and $a = d_2$, $c = d_1$, and $b = d_0$, in the notation of (1·2). There are three identities among the iterated face operators $G_2 \to G_0$; and each of the three squares whose commutativity express these equations, is in fact a pull-back square, with all four maps open surjections. The three identities thus give rise to six Beck–Chevalley conditions. We display for completeness all nine equations:

$$d_0 d_0 = d_0 d_1 \quad \left.\begin{array}{l} d_0^- d_0^+ = d_0^+ d_1^- \\ d_0^- d_0^+ = d_1^+ d_0^-, \end{array}\right\} \tag{1·9}$$

$$d_0 d_2 = d_1 d_0 \quad \left.\begin{array}{l} d_0^- d_1^+ = d_2^+ d_0^- \\ d_1^- d_0^+ = d_0^+ d_2^-, \end{array}\right\} \tag{1·10}$$

$$d_1 d_1 = d_1 d_2 \quad \left.\begin{array}{l} d_1^- d_1^+ = d_1^+ d_2^- \\ d_1^- d_1^+ = d_2^+ d_1^-. \end{array}\right\} \tag{1·11}$$

Note that (1·10) is (1·2), and $(1{\cdot}10)_1$ is (1·3).

Now to prove $\alpha^+\beta^- = \beta^+\alpha^-$, which the lemma claims, means in the simplicial notation to prove $d_1^+ d_0^- = d_0^+ d_1^-$. By symmetry it suffices to prove the inequality '$\leqslant$' here. This inequality is by adjointness equivalent to $d_0^- \leqslant d_1^- d_0^+ d_1^-$. We have

$$\begin{aligned} d_0^- &\leqslant d_0^- d_0^- d_0^+ && \text{(front adjunction)} \\ &= d_0^- d_0^+ d_1^- && \text{by } (1{\cdot}9)_1 \\ &= d_0^+ d_1^- d_1^- && \text{by } (1{\cdot}9)_1 \\ &= d_0^+ d_2^- d_1^- && \text{by } (1{\cdot}11) \\ &= d_1^- d_0^+ d_1^- && \text{by } (1{\cdot}10)_2, \end{aligned}$$

which is the desired inequality. This proves the lemma.

From Lemma 1·2, and Proposition 1·1 applied twice, we get the symmetric version of Proposition 1·1.

PROPOSITION 1·3. *Let* (1·1) *be an α-open localic groupoid. Then $\alpha^+\beta^- = \beta^+\alpha^-$ is a closure operator on $O(X)$, and its set $O(Q)$ of closed elements is the equalizer of α^- and β^-. In particular, the inclusion $q^-: O(Q) \hookrightarrow O(X)$ has a left adjoint q^+ which satisfies $q^-q^+ = \alpha^+\beta^- = \beta^+\alpha^-$. The locale map $q: X \to Q$ thus defined is the coequalizer of α and β; and q is an open surjection.*

Proof. Only the last assertion has yet to be established; we have to prove that q^+ satisfies the Frobenius reciprocity law $q^+(u \wedge q^- v) = q^+ u \wedge v$, but this is a simple calculation using $\alpha^- q^- = \beta^- q^-$ and Frobenius reciprocity for β^+.

COROLLARY. *If an α-open localic groupoid acts on a locale Y, then the action induces a closure operator 'orbit formation' on $O(Y)$.*

Proof. Apply Proposition 1·2 to the action groupoid $\mathbf{Y}$.

2. *A density theorem for localic categories*

We first remind the reader of the notion of a sublocale being strongly dense, introduced by Johnstone in [**5**]; for locales in **Sets**, it is equivalent to the standard density notion of [**4**], II.2·4. Recall that, for any locale X, we have a unique locale map $m: X \to 1$, thus a unique frame map $m^-: O(1) = \Omega \to O(X)$, which one conveniently omits from notation.

Definition ([5]). The locale map $i: R \to S$ has *strongly dense image* if for all $\theta \in \Omega$ and $w \in O(S)$, the inequality $i^{-}(w) \leqslant \theta$ implies $w \leqslant \theta$. If in addition i is an inclusion, R is called a *strongly dense sublocale* of S.

We consider now, as in §1, an α-open localic category $\alpha, \beta: R \rightrightarrows X$, and form, as there, the sub-coequalizer $q: X \to Q$ of α and β; then we form the sub-kernel-pair of q: by this we mean the comma square of q with itself

$$\begin{tikzcd} X \leqslant_Q X \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, phantom, "\leqslant"] & X \arrow[d, "q"] \\ X \arrow[r, "q"'] & Q. \end{tikzcd} \tag{2·1}$$

'Comma square' is a standard notion in 2-dimensional category theory, in particular in **Loc**, the category of locales, and for this category, an explicit construction is given in [11], §3. However, for the case at hand, a sub-kernel-pair for a q where q^{-} has a left adjoint q^{+}, the following alternative description is available. Recall (from [4], II.2, for example) that the product locale $X \times X$ has the property that $O(X \times X)$ is generated as a sup-lattice by 'open rectangles' $u \times v$, where u and $v \in O(X)$, and where

$$u \times v = \mathrm{proj}_0^{-}(u) \wedge \mathrm{proj}_1^{-}(v)$$

($\mathrm{proj}_i: X \times X \to X$ $(i = 0, 1)$ being the two projections). We consider the frame congruence $\equiv$ on $O(X \times X)$ generated by

$$u \times v \equiv (u \wedge \downarrow v) \times v, \tag{2·2}$$

where $\downarrow v = q^{-}q^{+}(v)$. We then define $X \leqslant_Q X \subseteq X \times X$ by letting $O(X \leqslant_Q X)$ be the quotient frame of $O(X \times X)$ modulo the frame congruence thus defined. Let $j: X \leqslant_Q X \hookrightarrow X \times X$ denote the inclusion. The locale maps a and $b: X \leqslant_Q X \to X$ are then just restrictions of proj_0 and $\mathrm{proj}_1: X \times X \to X$ along j. We do have the inequality asserted in (2·1), since if $v \in O(Q) \subseteq O(X)$, i.e. if $v = \downarrow v$, then

$$b^{-}(v) = j^{-}(1 \times v) = j^{-}(\downarrow v \times v) = j^{-}(v \times v) \leqslant j^{-}(v \times 1) = a^{-}(v),$$

the second equality sign by (2·2).

To check the universal property of $X \leqslant_Q X$ thus constructed, let $\alpha, \beta: Z \rightrightarrows X$ be arbitrary locale maps with $q\beta \leqslant q\alpha$; then, for any open rectangle $u \times v$,

$$\begin{aligned} \langle \alpha, \beta \rangle^{-}(u \times v) &= \alpha^{-}(u) \wedge \beta^{-}(v) = \alpha^{-}(u) \wedge \beta^{-}(\downarrow v) \wedge \beta^{-}(v) \\ &\leqslant \alpha^{-}(u) \wedge \alpha^{-}(\downarrow v) \wedge \beta^{-}(v) = \langle \alpha, \beta \rangle^{-}((u \wedge \downarrow v) \times v). \end{aligned}$$

The opposite inequality is obvious, so $\langle \alpha, \beta \rangle: Z \to X \times X$ factors through $X \leqslant_Q X$.

We may in particular apply this universal property of the sub-kernel-pair of q to the given localic category $\alpha, \beta: R \rightrightarrows X$, since $q\beta \leqslant q\alpha$ by construction of the sub-coequalizer q of β, α. So we get a comparison map $i: R \to X \leqslant_Q X$.

THEOREM 2·1. *Let $\alpha, \beta: R \rightrightarrows X$ be an α-open localic category. Then the comparison map i from R to the sub-kernel-pair $X \leqslant_Q X$ of the sub-coequalizer $q: X \to Q$ of β, α has strongly dense image.*

Proof. Let $w \in O(X \leqslant_Q X)$ satisfy $i^-(w) \leqslant \theta$ for some $\theta \in \Omega$. We must prove $w \leqslant \theta$. Since elements of the form $j^-(u \times v)$ generate $O(X \leqslant_Q X)$ as a sup-lattice, it suffices to consider the case $w = j^-(u \times v)$. So

$$\alpha^-(u) \wedge \beta^-(v) = i^- j^-(u \times v) \leqslant \theta = \alpha^-(\theta),$$

whence by adjointness $\alpha^+(\alpha^-(u) \wedge \beta^-(v)) \leqslant \theta$. We have, by Frobenius reciprocity and Proposition 1·1,

$$\alpha^+(\alpha^-(u) \wedge \beta^-(v)) = u \wedge \alpha^+\beta^-(v) = u \wedge {\downarrow} v,$$

so $u \wedge {\downarrow} v \leqslant \theta$. But then clearly $(u \wedge {\downarrow} v) \times v \leqslant \theta$, whence

$$j^-(u \times v) = j^-((u \wedge {\downarrow} v) \times v) \leqslant j^- \theta = \theta.$$

This proves the theorem.

We proceed to derive some corollaries.

THEOREM 2·2. *Let $\alpha, \beta : R \rightrightarrows X$ be an α-open localic groupoid. Then the comparison map i from R to the kernel pair $X \times_Q X$ of the coequalizer $q : X \to Q$ of α, β has strongly dense image.*

Proof. By Propositions 1·1 and 1·3, the coequalizer of α, β is the same as the sub-coequalizer, in this case. Furthermore the kernel pair $X \times_Q X$ is clearly a sublocale of the sub-kernel-pair $X \leqslant_Q X$, and i: $R \to X \leqslant_Q X$ factors through it, since $q\alpha = q\beta$. But i: $R \to X \leqslant_Q X$ has strongly dense image, by Theorem 2·1, hence so has its factorization through $X \times_Q X$.

COROLLARY 2·3. *Let $R \rightrightarrows X \times X$ be a localic equivalence relation with*

$$R \to X \times X \xrightarrow{\text{proj}_0} X$$

an open map. Then R is a strongly dense sublocale of $X \times_Q X$, where $Q = X/R$.

Proof. Consider the equivalence relation as a groupoid $R \rightrightarrows X$, and apply Theorem 2·2.

It is not in general true that $R = X \times_Q X$, as the following example shows. Let X be the set $\mathbb{N} \cup \{\infty\}$ with its natural linear ordering. Make it into a topological space by letting $O(X)$ consist of sets of form $[n, \infty] = \{k \mid n \leqslant k \leqslant \infty\}$, and $\varnothing$. This is a sober topological space. All open sets are compact, so in particular, X is coherent, by [**4**], II.3·4.

Let $R \subseteq X \times X$ be the subset consisting of (∞, ∞) and all (m, n), with $m - n$ an even number. Clearly R is an equivalence relation on X in the category of topological spaces. But the finite inverse limits needed to express this consists of coherent topological spaces and coherent maps (see [**4**], II.3·4). Moreover the natural functor O from the category of coherent spaces and coherent maps to locales preserves all limits, since via Stone Duality (see [**4**], II, corollary 3·4) it corresponds to the formation of free frame on a distributive lattice, which is left adjoint to the forgetful functor (see [**4**], II.2·11).

So we conclude that R is also an equivalence relation on X in the category of locales. We now argue that it is α-open. Consider any open subset V of R, say $V = R \cap U$ where $U \hookrightarrow X \times X$ is open. We must prove that $\text{proj}_0(V)$ is open in X, i.e.

is an upper set. Let $n \in \mathrm{proj}_0(V)$, so there exists m with $(n,m) \in R \cap U$. Let $n' \geqslant n$. We then have $(n', m) \in U$ and $(n', m+1) \in U$, since U is an upper set in both directions. But one of these two pairs must be in R, hence in V, and in either case, we get a witness that $n' \in \mathrm{proj}_0(V)$.

To calculate the locale quotient X/R, we just have to calculate the quotient space and then make it sober. The quotient space has three points, 'even', 'odd', and ∞, but it has chaotic topology. When sobrifying, we get the one point space. So the kernel pair of the quotient map is $X \times X$.

For finite locales (i.e. finite as frames), α-open equivalence relations are always kernel pairs; this will be discussed elsewhere.

3. *The Godement theorem, and related results*

Along with the notion of 'strongly dense' inclusion, Johnstone[**5**] introduced the notion of 'weakly closed' inclusion of locales. Again, for classical locale theory (i.e. locales in **Sets**), weakly closed is the same as closed; in general, it is weaker.

One has the following facts.

(1). An inclusion of locales which is weakly closed as well as strongly dense is an isomorphism. This is immediate from the construction of weak (= fibrewise) closure in [**5**], corollary 1·4.

(2). If a composite $A \to B \to C$ of inclusions is weakly closed, then so is $A \to B$. This is immediate from the pull-back stability of weak closedness (see [**5**], corollary 1·8).

We now have the following Godement theorem.

THEOREM A. *Any α-open equivalence relation R on X with $R \hookrightarrow X \times X$ weakly closed, is the kernel pair of its quotient map $X \to Q$.*

Proof. By Corollary 2·3, R is strongly dense in $X \times_Q X$, but it is also weakly closed there, by fact (2) above. Hence $R \cong X \times_Q X$, by fact (1) above.

We leave it to the reader to derive similarly from Proposition 1·1 an isomorphism result for α-open, weakly closed preorders R on $X: R \cong X \leqslant_Q X$.

Johnstone proved in [**5**] the following constructive version of the closed subgroup theorem of [**3**]: if $H \hookrightarrow G$ is a subgroup of a localic group G, and if $H \to 1$ is open, then H is weakly closed in G. Combining this with Theorem A, we can prove

THEOREM 3·1. *If H is a subgroup of a localic group G, and if $H \to 1$ is open, the kernel pair of the map $G \to G/H$ is given by the action of H. In particular, if H is normal (and $H \to 1$ is open), H is the kernel of the group homomorphism $G \to G/H$.*

Proof. The inclusion $i: H \times G \hookrightarrow G \times G$ given by $(h,g) \mapsto (hg, g)$ is an equivalence relation on G; and its β is the projection $H \times G \to G$ which is open since $H \to 1$ is open. Also there is a pull-back diagram

$$\begin{tikzcd} H \times G \arrow[r, hook, "i"] \arrow[d, "\mathrm{proj}"'] & G \times G \arrow[d, "n"] \\ H \arrow[r, hook] & G, \end{tikzcd}$$

where $n(g_1, g_2) = g_1 g_2^{-1}$, and since $H \hookrightarrow G$ is weakly closed, so is i, by the pull-back stability of this notion (see [**5**], corollary 1·8). From Theorem A it follows that $H \times G$

is the kernel pair of $G \to G/H$, which in view of the description of i yields the first assertion of the theorem. The second assertion is then derived by purely elementary properties of pull-back diagrams.

To prove Theorem B, we present a result which may have some independent interest. (It holds without the assumption of i being open; I thank Ieke Moerdijk for supplying me with a proof of this generalization.)

PROPOSITION 3·2. *Let $R' \overset{i}{\hookrightarrow} R \hookrightarrow X \times X$ be α-open equivalence relations on X, and assume i open. Let $u, v \in O(X)$. If $(u \times v) \cap R \subseteq R'$, then $(\bar{u} \times v) \cap R \subseteq R'$, where $\bar{u}$ is the saturation of u under the equivalence relation R'.*

Proof. Let $r \in O(R)$ define R', so that $i^- : O(R) \to O(R')$ is just $- \wedge r$. Then clearly saturation on $O(X)$ with respect to R' can be written $\bar{u} = d_1^+(r \wedge d_0^-(u))$ (where d_1 and d_0 are the α and $\beta : R \to X$, using the simplicial notation, as in the proof of Lemma 1·2). The assumption says that $d_1^-(u) \wedge d_0^-(v) \leqslant r$, so that

$$d_0^-(r) \geqslant d_0^-(d_1^-(u) \wedge d_0^-(v)) = d_2^- d_0^-(u) \wedge d_1^- d_0^-(v), \tag{3·1}$$

from (1·9) and (1·10). The transitivity of R' is expressed by the first inequality sign in

$$d_1^-(r) \geqslant d_2^-(r) \wedge d_0^-(r) \geqslant d_2^-(r) \wedge d_2^- d_0^-(u) \wedge d_1^- d_0^-(v),$$

the second inequality sign by (3·1). By adjointness, we therefore get the inequality

$$\begin{aligned} r &\geqslant d_1^+(d_2^-(r) \wedge d_2^- d_0^-(u) \wedge d_1^- d_0^-(v)) = d_1^+(d_2^-(r) \wedge d_2^- d_0^-(u)) \wedge d_0^-(v) \\ &= d_1^+(d_2^-(r \wedge d_0^-(u))) \wedge d_0^-(v) = d_1^-(d_1^+(r \wedge d_0^-(u)) \wedge d_0^-(v) = \bar{u} \times v, \end{aligned}$$

where we used Frobenius and $(1·11)_1$.

We can use this Proposition to prove a sharpened version of Moerdijk's Theorem B, mentioned in the introduction:

THEOREM B′. *Let $R' \overset{i}{\hookrightarrow} R \overset{j}{\hookrightarrow} X \times X$ be two α-open equivalence relations on X. If they have the same quotient, and if i is open, then i is an isomorphism.*

Proof. The idea of the following proof is essentially from Moerdijk [**9**]. Let $Q = X/R = X/R'$. We may then consider all diagrams involved as diagrams in **Loc**$/Q$, the category of locales over Q, which by [**6**] is equivalent to the category of all locales in the topos sh (Q) of sheaves on Q. So, changing to the topos sh (Q), we may assume that $Q = 1$, and that $q : X \to 1$ is an open surjection. Then $O(X \times X)$ is generated as a sup-lattice by rectangles $u \times v$, with $u, v \in O(X)$ satisfying

$$q^+(u) = q^+(v) = 1 \tag{3·2}$$

(essentially by lemma V·5·2 in [**6**]). So $O(R)$ is generated by elements $(u \times v) \cap R$ with u, v satisfying (3·2). Since $R' \to 1$ is surjective and since $R' \hookrightarrow R$ is open, some such $(u \times v) \cap R$ is contained in R'. So take one. Applying Proposition 3·2 to it twice we conclude that $(\bar{u} \times \bar{v}) \cap R \subseteq R'$. But

$$\bar{u} = q^- q^+(u) = q^-(1) = X,$$

by (3·2), and similarly $\bar{v} = X$. So $(X \times X) \cap R \subseteq R'$, and thus $R = R'$.

We now prove Theorem B as stated in the introduction. Openness of $R \hookrightarrow X \times X$ and of $X \to 1$ imply that R is α-open. Moreover openness of R in $X \times X$ implies openness of R in $X \times_Q X$, and Theorem B′ can then be applied to $i : R \hookrightarrow X \times_Q X$ to yield that it is an isomorphism.

To state our final theorem, let us say that a sublocale $R \hookrightarrow S$ is *weakly locally closed* if it is open in its weak closure. Then we can combine Theorems A and B′ to get the following result; I am indebted to Ieke Moerdijk for pointing out an error in my first version of this ([7], theorem 8).

THEOREM 3·3. *If $R \hookrightarrow X \times X$ is a weakly locally closed equivalence relation, with weak closure $\bar{R}$ of R α-open, and with $X \times_Q X \hookrightarrow X \times X$ weakly closed (where $Q = X/R$), then $R = \bar{R} = X \times_Q X$.*

Proof. Since $X \times_Q X$ is weakly closed, $\bar{R}$ is contained in $X \times_Q X$, and so R and $\bar{R}$ have the same quotient Q. By Theorem B′, $R = \bar{R}$. So R is weakly closed in $X \times X$, and hence by Theorem A, $R = X \times_Q X$.

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