

Fibrations as Eilenberg-Moore algebras

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Introduction

We give an elementary account of some fundamental facts about fibered (or rather opfibered) categories, in terms of monads and 2-categories. The account avoids any mention of category-valued functors and pseudofunctors.

I make no claim to originality; a large part of it may in substance be found in the early French literature on the subject (early 1960s - Grothendieck, Chevalley, Giraud, Benabou), as reported in [Gray 1966]. The remaining part makes use of some tools that were not fully available then: monads, and their algebras, and KZ monads; it is extracted from [Kock 1973, 1995] and [Street 1974]. Some more historical comments are found in the last Section.

1 Cocartesian arrows

Given a functor $\pi : \mathcal{X} \rightarrow \mathcal{B}$. For $a : A \rightarrow B$ in \mathcal{B} , we let $\text{hom}_a^{(\mathcal{X}, \pi)}$ denote the set of arrows x in \mathcal{X} with $\pi(x) = a$; if $\pi : \mathcal{X} \rightarrow \mathcal{B}$ is clear from the context, we write just hom_a . We fix a $\pi : \mathcal{X} \rightarrow \mathcal{B}$ in this Section.

If a is the identity arrow of A , $\text{hom}_a^{\mathcal{X}}$ is also written $\text{hom}_A^{\mathcal{X}}$; it consists of the *vertical* arrows of \mathcal{X} over A (relative to $\pi : \mathcal{X} \rightarrow \mathcal{B}$)¹. We denote by \mathcal{X}_A the category whose objects are the objects $X \in \mathcal{X}$ with $\pi(X) = A$, and whose arrows are the vertical arrows over A . It is a (non-full) subcategory of \mathcal{X} , often called the fibre over A .

¹Sometimes, one needs to say “ π -vertical” rather than just “vertical”, namely in contexts where one also wants to talk about “vertical arrows” meaning arrows displayed vertically in the graphics of a certain diagram. Often in diagrams, one likes to display π -vertical arrows by graphically vertical arrows.

Given arrows $x : X \rightarrow Y$ in and $y : Y \rightarrow Z$ in \mathcal{X} . Let a and b denote $\pi(x)$ and $\pi(y)$, respectively, thus $x \in \text{hom}_a(X, Y)$ and $y \in \text{hom}_b(Y, Z)$. Then a and b are composable in \mathcal{B} , and $x.y \in \text{hom}_{a.b}(X, Z)$ ². Thus, for fixed $x \in \text{hom}_a(X, Y)$, we have a map “precomposition with x ”,

$$x^* : \text{hom}_b(Y, Z) \rightarrow \text{hom}_{a.b}(X, Z). \quad (1)$$

The following notion is the crucial one for the presentation here.

Definition 1.1 *The arrow $x \in \text{hom}_a(X, Y)$ is cocartesian³ if for all arrows b in \mathcal{B} with domain $\pi(Y)$ and all $Z \in \mathcal{X}_C$ (where C denotes the codomain of b), the map (1) is bijective.*

Note that the only “data” in the definition (besides $\pi : \mathcal{X} \rightarrow \mathcal{B}$) is the arrow x . To keep track of the “book-keeping” involved, we display a diagram, in which the symbol “:” is meant to indicate “goes by π to ...”. The “data” a , A , and B are derived from x (with $a = \pi(x)$, and A and B the domain and codomain of a); and b, C and Z are arbitrary.

$$\begin{array}{ccc} X & \xrightarrow{x} & Y & & Z \\ & & : & & : \\ & & : & & : \\ & & : & & : \end{array} \quad (2)$$

$$A \xrightarrow{a} B \xrightarrow{b} C.$$

Another way of describing when an arrow x (over a , say) is cocartesian is to say that it has a certain (co-)universal property: for any arrow in \mathcal{X} with same domain as x and living over a composite arrow of the form $a.b$, factorizes as $x.y$ for a unique y over b . This is reminiscent of the (co-)universal property of a

²We compose arrows in an abstract category from left to right, $a.b$ means “first a , then b ”; whereas we compose functors between given categories from right to left, thus $G \circ F$ means “first F , then G ”.

³or op-cartesian, or, cf. [Johnstone 2002], supine. Note that in the classical definition ([Giraud 1971] p. 18) “(co-)cartesian morphism” means something weaker, namely as above, but with b an identity arrow; an op-fibered category is then defined as one where there are enough of these “weakly” (co-)cartesian arrows, and where such arrows compose. In this case, the weak and strong notions coincide. Thus, in the set up of loc.cit., the notion of op-fibered category is needed prior to the definition of coartesian arrow, in the strong sense as given above. See [Borceux 1994] 8.1 for a comparison of the weak (“pre-”) and the strong notion.

coequalizer x : “any arrow with same domain as x (and with a certain property) factors uniquely over x .”

We have the following, in case the composite $x.y$ in \mathcal{X} is defined:

Proposition 1.2 *Suppose x cocartesian. Then y is cocartesian iff $x.y$ is cocartesian.*

Proof. Straightforward verification; or see [Borceux 1994] Section 8.1.

We may read the bijections in (1) as a universal property of cocartesian arrows x . Using this viewpoint, one gets

Proposition 1.3 *If $x : X \rightarrow Y$ and $x' : X \rightarrow Y'$ are cocartesian arrows with $\pi(x) = \pi(x')$, then there exists a unique vertical isomorphism $t : Y \rightarrow Y'$ with $x.t = x'$. Conversely, if $x : X \rightarrow Y$ is cocartesian, and $t : Y \rightarrow Y'$ is a vertical isomorphism, then $x.t$ is cocartesian. A vertical arrow is cocartesian iff it is invertible.*

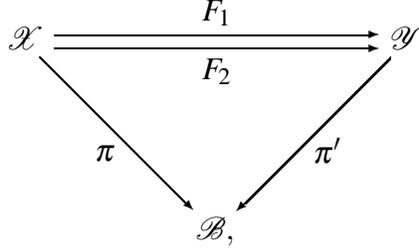
2 The 2-category Cat/\mathcal{B}

Let \mathcal{B} be a category. The objects of Cat/\mathcal{B} are the functors with codomain \mathcal{B} , like the $\pi : \mathcal{X} \rightarrow \mathcal{B}$ considered in Section 1. The morphisms are the strictly commutative triangles (“functors over \mathcal{B} ”)

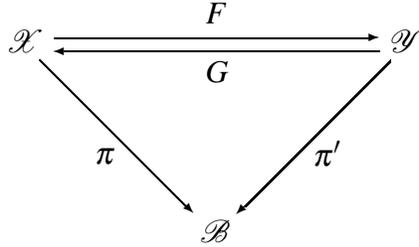
$$\begin{array}{ccc}
 \mathcal{X} & \xrightarrow{F} & \mathcal{Y} \\
 & \searrow \pi & \swarrow \pi' \\
 & \mathcal{B} &
 \end{array}
 \tag{3}$$

thus Cat/\mathcal{B} is a standard slice category. Note that since the triangle commutes (strictly), F preserves the property for arrows of “being over a ”, and in particular preserves the property of being vertical. To make the slice category Cat/\mathcal{B} into a (strict) 2-category, we describe what are the 2-cells:

Given two parallel morphisms in Cat/\mathcal{B} , as displayed in



the 2-cells between these are taken to be the natural transformations $\tau : F_1 \Rightarrow F_2$ which are *vertical*, meaning that for all objects $X \in \mathcal{X}$, τ_X is a vertical arrow in \mathcal{Y} . Composition (horizontal as well as vertical) is inherited from the standard composition of natural transformations in Cat . – Having a 2-category, one has a notion of *adjoint* arrows “ $F \dashv G$ by virtue of 2-cells η, ε ”. In particular, for the 2-category Cat/\mathcal{B} , adjointness of two arrows (functors over \mathcal{B})



amounts to an ordinary adjointness η, ε between the functors F and G , subject to the further requirement that η and ε are *vertical* natural transformations. In this case, one may write $F \dashv_{\mathcal{B}} G$. If $F \dashv_{\mathcal{B}} G$, then the bijection, due to $F \dashv G$, between $\text{hom}(F(X), Y)$ and $\text{hom}(X, G(Y))$, restricts to a bijection

$$\text{hom}_a(F(X), Y) \cong \text{hom}_a(X, G(Y))$$

whenever $a : A \rightarrow B$, $X \in \mathcal{X}_A$, $Y \in \mathcal{Y}_B$; for, let $f : F(X) \rightarrow Y$ be an arrow in \mathcal{Y} over a . The arrow $X \rightarrow G(Y)$ corresponding to it under the adjointness is the composite $\eta_X \cdot G(f)$, which an arrow over a since $G(f)$ is so, and since η_X is vertical, by assumption on η . Similarly, verticality of ε_Y proves that the inverse correspondence preserves the property of being over a .

Proposition 2.1 (Key Lemma) *Let F and G be vertically adjoint, $F \dashv_{\mathcal{B}} G$, as above. Then F preserves the property of being cocartesian.*

Proof. Let $x : X \rightarrow Y$ in \mathcal{X} be cocartesian over $a : A \rightarrow B$, and let $b : B \rightarrow C$ be an arbitrary arrow in \mathcal{B} . For any $Z \in \mathcal{X}_C$, the standard naturality square for hom set bijections induced by the ordinary adjointness $F \dashv G$ restricts to a commutative square

$$\begin{array}{ccc}
 \mathrm{hom}_b(Y, G(Z)) & \xrightarrow{\cong} & \mathrm{hom}_b(F(Y), Z) \\
 \downarrow x^* & & \downarrow F(x)^* \\
 \mathrm{hom}_{a.b}(X, G(Z)) & \xrightarrow{\cong} & \mathrm{hom}_{a.b}(F(X), Z).
 \end{array}$$

The hom set bijections are displayed horizontally. The left hand vertical map is a bijection since x is cocartesian. Hence so is the right hand vertical map; so $F(x)$ is cocartesian.

3 Opfibrations; cleavages and splittings

Definition 3.1 *Given a functor $\mathcal{X} \rightarrow \mathcal{B}$. It is called an opfibration if it has enough cocartesian arrows, in the sense that for any arrow a in \mathcal{B} and any $X \in \mathcal{X}$ with $\pi(X) = d_0(a)$, there exists a cocartesian arrow over a with domain X .*

A cocartesian arrow x over a is called a “cocartesian lift” of a . It is a cocartesian lift of a from X if furthermore its domain is X .

For the remainder of the present Section, $\pi : \mathcal{X} \rightarrow \mathcal{B}$ is assumed to be an opfibration.

Proposition 3.2 *Given an arrow z in \mathcal{X} with $\pi(z) = a.b$ for arrows a and b in \mathcal{B} . Then z may be factorized $x.y$ with $\pi(x) = a$ and $\pi(y) = b$, with x cocartesian. This factorization is unique modulo a unique vertical isomorphism in the middle. And y is cocartesian iff z is cocartesian.*

Proof. Let x be a cocartesian lift of a with same domain as z , and construct y over b , with $x.y = z$, using the universal property of x . – The last assertion now follows from Proposition 1.2.

A special case is (take b to be the relevant identity arrow):

Proposition 3.3 *Every arrow z in \mathcal{X} may be factored into a cocartesian followed by a vertical arrow. This factorization is unique modulo a unique vertical isomorphism.*

Definition 3.4 *A cleavage for $\mathcal{X} \rightarrow \mathcal{B}$ consists in a choice $X \triangleright a$ of a cocartesian lift of a from X , for every X and for every a with $\pi(X) = d_0(a)$.*

If there is given a cleavage \triangleright , it is convenient to have a separate notation for the codomain of the chosen cocartesian arrow $X \triangleright a$; common notations are $a_!(X)$, $\Sigma_a(X)$ or $\exists_a(X)$, or, the one we shall use, $a_*(X) := d_1(X \triangleright a)$,

$$X \xrightarrow{X \triangleright a} a_*(X). \quad (4)$$

If a cleavage is given, the uniqueness assertions “modulo vertical isomorphisms” in the Propositions 3.2 and 3.3 may be sharpened to strict uniqueness, by requiring the cocartesian arrows to be provided by the cleavage, in an evident way, thus for Proposition 3.3: $z : X \rightarrow Y$ factors uniquely as $X \triangleright \pi(z)$ followed by a vertical arrow.

There are dual notions: cartesian arrows, cartesian lifts, fibrations, with associated cleavage/splitting terminology. By experience, they are more important than opfibrations. The reason we discuss opfibrations rather than fibrations is that they are more straightforward in so far as variance is concerned. Otherwise, the mathematics is the same. Let us right away describe our notation, corresponding to (4), for a cleavage of a fibration:

$$a^*(X) \xrightarrow{a \triangleleft X} X \quad (5)$$

where now $\pi(X) = d_1(a)$. (In Giraud’s notation: $X^a \xrightarrow{X_a} X$.)

Consider a functor $F : \mathcal{X} \rightarrow \mathcal{Y}$ over \mathcal{B} , as displayed in (3). Assume both $\mathcal{X} \rightarrow \mathcal{B}$ and $\mathcal{Y} \rightarrow \mathcal{B}$ are opfibrations. Then the F is called a *morphism of opfibrations* if it takes cocartesian arrows to cocartesian arrows.

Note that both being an opfibration and being a morphism of opfibrations are *properties* of categories (resp. functors) over \mathcal{B} , not a *structure* that is given (resp. is preserved). In contrast, a cleavage is a structure, and morphisms of opfibrations may or may not preserve cleavages.

If the chosen lifts of identity arrows are identity arrows, the cleavage is called *normalized*; thus, if $\pi(X) = A$, we have

$$X \triangleright 1_A = 1_X.$$

If furthermore the composite of two chosen cocartesian arrows is again chosen, the cleavage is called a *splitting*:

$$(X \triangleright a).(a_*(X) \triangleright b) = X \triangleright (a.b). \quad (6)$$

Taking the codomain of the two sides of this equation gives in particular that

$$b_*(a_*(X)) = (a.b)_*(X). \quad (7)$$

Some of the literature on (op-) fibrations formulate the theory of (op-) fibrations mainly in terms of a cleavages/splittings – which in turn can be reformulated in terms of *Cat*-valued pseudofunctors/functors.

Example. A group may be considered as a category, in which there is only one object, and where all arrows are invertible. If \mathcal{X} and \mathcal{B} are groups, a functor $\pi : \mathcal{X} \rightarrow \mathcal{B}$ is the same as a group homomorphism. The vertical arrows form the kernel of π . All arrows in \mathcal{X} are cocartesian. A group homomorphism $\pi : \mathcal{X} \rightarrow \mathcal{B}$ is an opfibration iff it is surjective. A (normalized) cleavage is a (set theoretical) section s of π (taking the identity arrow of \mathcal{B} to the identity arrow of \mathcal{X}). Then s is a splitting iff s is a group homomorphism. Not every surjective group homomorphism admits such a splitting. So there are opfibrations which do not admit splittings.

Remark. Any opfibration may, by the axiom of choice, be supplied with a normalized cleavage; but not necessarily with a splitting, as the example shows. However, every opfibered category $\mathcal{X} \rightarrow \mathcal{B}$ is *equivalent* in the 2-category Cat/\mathcal{B} to (the underlying opfibration of) a split opfibration over \mathcal{B} , see Section 7 below.

This does not contradict the fact just mentioned about non-existence of splittings of surjective group homomorphism. For, a category equivalent to a group need not be a group; it may have several objects.

4 The “opfibration” monad on Cat/\mathcal{B}

Given an object in Cat/\mathcal{B} , i.e. a functor $\pi : \mathcal{X} \rightarrow \mathcal{B}$. One derives from this a new $T(\pi : \mathcal{X} \rightarrow \mathcal{B}) \in Cat/\mathcal{B}$: it is the comma category $\pi \downarrow \mathcal{B}$, equipped with the “codomain” functor to \mathcal{B} . Recall that an object of $\pi \downarrow \mathcal{B}$ is a pair (X, a) , where $X \in \mathcal{X}$ and a is an arrow in \mathcal{B} with domain $\pi(X)$. The codomain functor $d_1 : \pi \downarrow \mathcal{B} \rightarrow \mathcal{B}$ takes this object to the codomain of a . The arrows in Cat/\mathcal{B} are given by arrows in \mathcal{X} together with suitable commutative squares in \mathcal{B} ; for a display, see e.g. (11) below.

may be depicted

$$\begin{array}{ccc}
 X & & \\
 \searrow x & & \\
 & & X' \\
 : & & \\
 A & \xrightarrow{a} & B \\
 \searrow \pi(x) & & \searrow \beta \\
 & & A' \xrightarrow{a'} B'
 \end{array}
 \tag{11}$$

(with the bottom square commutative), and it may be denoted $(x, \beta) : (X, a) \rightarrow (X', a')$; it is an arrow over β .

The following shows that $T(\mathcal{X} \rightarrow \mathcal{B})$, as a category over \mathcal{B} , has some canonical cocartesian arrows:

Proposition 4.1 *Given an object (X, a) in $T(\mathcal{X} \rightarrow \mathcal{B})$ over B , and given an arrow $b : B \rightarrow C$ in \mathcal{B} . Then there is a canonical cocartesian arrow over b from (X, a) to $(X, a.b)$, depicted in*

$$\begin{array}{ccc}
 X & & \\
 \searrow 1_X & & \\
 & & X \\
 : & & \\
 A & \xrightarrow{a} & B \\
 \searrow 1_A & & \searrow b \\
 & & A \xrightarrow{a.b} C
 \end{array}
 \tag{12}$$

We denote this arrow in $T(\mathcal{X} \rightarrow \mathcal{B})$ by $((X, a); b)$, it is an arrow over b from (X, a) to $(X, a.b)$. We may define a cleavage by putting $(X, a) \triangleright b := ((X, a); b)$, thus

$$(X, a) \xrightarrow{(X, a) \triangleright b} (X, a.b).$$

Proof. To see that the depicted arrow $((X, a); b) : (X, a) \rightarrow (X, a.b)$ is cocartesian over $b : B \rightarrow C$, let $c : C \rightarrow D$ and let (Z, δ) be an object over D (where $\delta : D' \rightarrow D$ and $\pi(Z) = D'$). Then as subsets of $\text{hom}(X, Z)$, we see that $\text{hom}_c((X, a.b), (Z, \delta))$ consists of those $h : X \rightarrow Z$ which satisfy $\pi(h) \cdot \delta = (a.b) \cdot c$, and $\text{hom}_{b.c}((X, a), (Z, \delta))$ of those $h : X \rightarrow Z$ which satisfy $\pi(h) \cdot \delta = a \cdot (b.c)$, and these two subsets are equal. Precomposition with (12) is provided by precomposition with the \mathcal{X} component which here is 1_X , so is indeed the identity mapping of the described subset onto itself.

From the Proposition immediately follows that $T(\mathcal{X} \rightarrow \mathcal{B}) \rightarrow \mathcal{B}$ has sufficiently many cocartesian arrows to deserve the title ‘‘opfibration’’, and in fact, the very construction of the canonical cocartesian arrows shows that it provides this opfibration with a *splitting*: the codomain of $((X, a); b)$ is $(X, a.b)$, and clearly

$$((X, a); b) \cdot ((X, a.b); c) = ((X, a); b.c).$$

The functorial character of T is straightforward from the construction (8). Explicitly: for a functor over \mathcal{B} , as depicted in (3), $T(F)$ is the functor $T(\mathcal{X} \rightarrow \mathcal{B}) \rightarrow T(\mathcal{Y} \rightarrow \mathcal{B})$ which on objects is given by $T(F)(X, \alpha) = (F(X), \alpha)$, and on arrows $T(F)(x) = F(x)$ (if x satisfies the equation (10), then so does $F(x)$ (with π replaced by π')).

We make T into a monad by supplying natural transformations $y : Id \Rightarrow T$ and $m : T^2 \Rightarrow T$. We describe first the objects of $T^n(\mathcal{X} \rightarrow \mathcal{B})$: an object over $A \in \mathcal{B}$ of $T^n(\mathcal{X} \rightarrow \mathcal{B})$ may be depicted

$$\begin{array}{c} X \\ \\ : \\ \\ A^{(n)} \longrightarrow \dots \longrightarrow A'' \longrightarrow A' \longrightarrow A. \end{array}$$

The unit y and multiplication m of the monad come about from the units and the composition in the chain of $A^{(i)}$ s (and this makes the unit and associative

laws for y and m evident). Thus (if we suppress π from notation), the functor $y_{\mathcal{X}} : \mathcal{X} \rightarrow T(\mathcal{X})$ takes an $X \in \mathcal{X}_A$ to the configuration

$$\begin{array}{c} X \\ \\ : \\ \\ A \xrightarrow{1_A} A, \end{array}$$

or $X \mapsto (X, 1_A)$. Similarly, the object $(X, a, b) \in T^2(\mathcal{X})$ goes by $\mu_{\mathcal{X}}$ to the object $(X, a, b) \in T(\mathcal{X})$.

Consider an object $(X, a) \in T(\mathcal{X})$, as depicted in (9); then $T(y_{\mathcal{X}})(X, a)$ is the object $(X, 1, a)$, depicted in

$$\begin{array}{c} X \\ \\ : \\ \\ A \xrightarrow{1_A} A \xrightarrow{a} B, \end{array} \tag{13}$$

and $y_{T(\mathcal{X})}(X, a)$ is the object $(X, a, 1)$ depicted in

$$\begin{array}{c} X \\ \\ : \\ \\ A \xrightarrow{a} B \xrightarrow{1_B} B, \end{array} \tag{14}$$

The reader may, as an exercise, describe a vertical arrow in $T^2(\mathcal{X})$ from the first of these objects to the second.

The endofunctor T on Cat/\mathcal{B} is clearly canonically enriched over Cat : its value on a 2-cell between 1-cells F and G , i.e. on a vertical natural transformation $t : F \Rightarrow G$, is the vertical natural transformation whose instantiation at $(X, a) \in T(\mathcal{X})$ (where $a : A \rightarrow B$) associates $t_X : F(X) \rightarrow G(X)$ (it may be viewed as an element in $\text{hom}_A^{\mathcal{Y}}((F(X), a)(G(X), a))$ since it satisfies an equation like (10): $\pi'(t_X).a = a.1_A$, because $\pi'(t_X)$ is an identity arrow). And y and m are 2-natural.

The name ‘‘opfibration monad’’ should really be, more precisely: the ‘‘split-opfibration monad’’; we have already argued that every $T(\mathcal{X} \rightarrow \mathcal{B})$ has canonically the structure of a *split* opfibration; we shall see that $T(\mathcal{X} \rightarrow \mathcal{B})$ is the free such on $\mathcal{X} \rightarrow \mathcal{B}$, in fact, we shall see that the category of Eilenberg-Moore algebras for this monad is the category of split opfibrations over \mathcal{B} .

5 KZ aspects

The monad (T, y, m) on Cat/\mathcal{B} has now been enhanced to a (strict) 2-monad. It can be even further enhanced, namely to a KZ monad in the sense of [Kock 1973, 1995]. This means that it is provided with a modification $\lambda : T(y) \Rightarrow y_T$, with certain equational properties. Concretely, it here means that for each object $\mathcal{X} = (\pi : \mathcal{X} \rightarrow \mathcal{B})$ in Cat/\mathcal{B} , there is provided a vertical natural transformation $\lambda = \lambda_{\mathcal{X}}$,

$$T(\mathcal{X}) \begin{array}{c} \xrightarrow{T(y_{\mathcal{X}})} \\ \Downarrow \lambda \\ \xrightarrow{y_{T(\mathcal{X})}} \end{array} T^2(\mathcal{X})$$

satisfying two ‘‘whiskering’’ equations (16) and (17) below. Consider an object $(X, a) \in T(\mathcal{X})$, as in (9); we describe $\lambda_{(X,a)}$. Recall the description of the objects $T(y_{\mathcal{X}})(X, a)$ and $y_{T(\mathcal{X})}(X, a)$ given in (13) and (14), respectively. Then $\lambda_{(X,a)}$ is the arrow in $T^2(\mathcal{X})$ given by the following. The unnamed arrows are identity arrows.

$$\begin{array}{c}
 X \\
 \searrow \\
 X \\
 : \\
 \begin{array}{ccccc}
 A & \xrightarrow{\quad} & A & \xrightarrow{a} & B \\
 & \searrow & \searrow a & & \searrow \\
 & & A & \xrightarrow{a} & B & \xrightarrow{\quad} & B
 \end{array}
 \end{array}
 \tag{15}$$

(Cf. the Exercise comparing (13) and (14)). It is clear that if a is an identity arrow in \mathcal{B} , then $\lambda_{(X,a)}$ is an identity arrow in $T(\mathcal{X})$, which implies that the whiskering

$$\mathcal{X} \xrightarrow{y_{\mathcal{X}}} T(\mathcal{X}) \xrightarrow[\underset{y_{T(\mathcal{X})}}{\Downarrow \lambda}]{T(y_{\mathcal{X}})} T^2(\mathcal{X}) \quad (16)$$

is an identity natural transformation. Also, applying $m_{\mathcal{X}}$ to the arrow in $T^2(\mathcal{X})$ exhibited in (15) produces the identity arrow of (X, a) in $T(\mathcal{X})$, and this implies that the whiskering

$$T(\mathcal{X}) \xrightarrow[\underset{y_{T(\mathcal{X})}}{\Downarrow \lambda}]{T(y_{\mathcal{X}})} T^2(\mathcal{X}) \xrightarrow{m_{\mathcal{X}}} T(\mathcal{X}) \quad (17)$$

is the identity natural transformation. These two whiskering equations are what a modification λ should satisfy in order to make a (strict) monad on a 2-category into a KZ monad, cf. [Kock 1995], Axioms T0-T3 (with T0 and T3 being redundant if (T, y, m) is a strict monad, which is the case here).

Whenever one has a monad T on a category \mathcal{C} , one has the category of (Eilenberg-Moore) algebras for it, i.e. an object $\mathcal{X} \in \mathcal{C}$ together with a “structure” map $\xi : T(\mathcal{X}) \rightarrow \mathcal{X}$, satisfying the standard unit- and associativity equations.

Recall that in a 2-category, the notion of adjointness between 1-cells makes sense.

We denote objects in a 2-category \mathcal{C} by script letters like \mathcal{X} , because of the example we have in mind. Also, we compose the arrows in \mathcal{C} from right to left.

Here is a basic construction in the context of KZ monads (cf. [Kock 1973, 1995]). To produce an adjointness $\xi \dashv y_{\mathcal{X}}$ out of an Eilenberg-Moore algebra \mathcal{X}, ξ , we produce unit η and counit ε . For the counit ε , we just take the identity 2-cell on $\xi \circ y_{\mathcal{X}} = id_{\mathcal{X}}$. The unit η is constructed using λ : we have the 2-cell obtained by whiskering $\lambda_{\mathcal{X}}$ with $T(\xi)$:

$$T(\mathcal{X}) \xrightarrow[\underset{y_{T(\mathcal{X})}}{\Downarrow \lambda}]{T(y_{\mathcal{X}})} T^2(\mathcal{X}) \xrightarrow{T(\xi)} T(\mathcal{X}). \quad (18)$$

The top composite is an identity 1-cell, since $\xi \circ y_{\mathcal{X}} = id_{\mathcal{X}}$. The lower composite may be rewritten as $y_{\mathcal{X}} \circ \xi$, using naturality of y w.r.to ξ . So the whiskering (19)

gives a 2-cell

$$T(\mathcal{X}) \begin{array}{c} \xrightarrow{id} \\ \Downarrow \\ \xrightarrow{y_{\mathcal{X}} \circ \xi} \end{array} T(\mathcal{X}), \quad (19)$$

and this is the unit η of the adjointness $\xi \dashv y_{\mathcal{X}}$. The triangle equation holds by virtue one of the whiskering equation (16). In particular, since $m_{\mathcal{X}}$ is a T -homomorphism, we have $m_{\mathcal{X}} \dashv y_{T(\mathcal{X})}$.

If $\xi \dashv y_{\mathcal{X}}$, it does not conversely follow that ξ is an Eilenberg-Moore algebra structure, since the associative law $\xi \circ T(\xi) = \xi \circ m$ may not hold strictly; but we have from [Kock 1973, 1995]

Proposition 5.1 *If T is a KZ monad, and if $\xi : T(\mathcal{X}) \rightarrow \mathcal{X}$ is a left adjoint for $y_{\mathcal{X}}$, then there is a canonical isomorphism (2-cell) α between the two 1-cells $\xi \circ T(\xi)$ and $\xi \circ m_{\mathcal{X}}$.*

Proof. Since $\xi \dashv y_{\mathcal{X}}$, and T is a 2-functor, it follows that $T(\xi) \dashv T(y_{\mathcal{X}})$. Also, since $m_{\mathcal{X}} : T^2(\mathcal{X}) \rightarrow T(\mathcal{X})$ is an Eilenberg-Moore structure by general monad theory, and because T is KZ, it follows that $m_{\mathcal{X}} \dashv y_{T(\mathcal{X})}$. We therefore have that

$$\xi \circ T(\xi) \dashv T(y_{\mathcal{X}}) \circ y_{\mathcal{X}}$$

and partly that

$$\xi \circ m_{\mathcal{X}} \dashv y_{T(\mathcal{X})} \circ y_{\mathcal{X}};$$

but the two right hand sides here are equal, by naturality of y , so it follows that the two left adjoints exhibited are canonically isomorphic.

For a KZ monad, it can be proved (cf. loc.cit) that the canonical isomorphism α satisfies those coherence equations which make \mathcal{X}, ξ into an Eilenberg-Moore pseudo-algebra, in the standard sense of 2-dimensional monad theory; vice versa, a pseudo-algebra \mathcal{X}, ξ, α in the standard sense has $\xi \dashv y_{\mathcal{X}}$.

6 Cleavages and splittings in terms of the opfibration monad

We now return to the case where $\mathcal{C} = \text{Cat}/\mathcal{B}$, and where T is the ‘‘opfibration KZ monad’’ described. Recall that \mathcal{X} is also used as a shorthand for an object $\pi : \mathcal{X} \rightarrow \mathcal{B}$ in Cat/\mathcal{X} .

Theorem 6.1 1) Assume $\xi : T(\mathcal{X}) \rightarrow \mathcal{X}$ is vertically left adjoint to $y_{\mathcal{X}}$ in Cat/\mathcal{X} , and with the identity functor on \mathcal{X} as counit (so $\xi \circ y_{\mathcal{X}} = id_{\mathcal{X}}$).⁴ Then \mathcal{X} carries a canonical structure of normalized cleavage. Conversely, a normalized cleavage defines a functor ξ , left adjoint to $y_{\mathcal{X}}$ and with $\xi \circ y_{\mathcal{X}} = id$.

2) This normalized cleavage is a splitting if and only if ξ is strictly associative, $T(\xi) \circ \xi = m \circ \xi$.

3) If (\mathcal{X}, ξ) and (\mathcal{X}', ξ') are strict T -algebras, then a functor $\mathcal{X} \rightarrow \mathcal{X}'$ over \mathcal{B} is a T -homomorphism iff it preserves the corresponding splittings strictly.

Proof/Construction. Given ξ . Let $X \in \mathcal{X}_A$, and let $a : A \rightarrow B$ in \mathcal{B} . We produce our candidate for a cocartesian lift $X \triangleright a : X \rightarrow a_*(X)$ by applying ξ to the arrow $((X, 1_A); a)$ in $T(\mathcal{X})$ exhibited in the following

$$\begin{array}{ccc}
 X & & \\
 \searrow^{1_X} & & \\
 & X & \\
 : & & \\
 A & \xrightarrow{1_A} & A \\
 \searrow^{1_A} & & \searrow^a \\
 & A & \xrightarrow{a} & B
 \end{array} \tag{20}$$

Thus $a_*(X)$ is $\xi(X, a)$. Note that (20) is a special case of the general canonical cocartesian arrow (12) in $T(\mathcal{X})$. So (20) is cocartesian, and therefore, applying ξ to it gives, by the Key Lemma (Proposition 2.1), a cocartesian arrow in \mathcal{X} . Its domain is X , since the top line in the (20) is $y_{\mathcal{X}}(X)$ and $\xi \circ y_{\mathcal{X}} = id_{\mathcal{X}}$. It lives over a , since the right hand slanted arrow in (20) is a . Thus we have constructed a cleavage for \mathcal{X} . The fact that the constructed cleavage is normalized follows because $\xi \circ y_{\mathcal{X}}$ is the identity functor, and because ξ , as a functor, takes identity arrows to identity arrows.

Conversely, given a normalized cleavage \triangleright of $\mathcal{X} \rightarrow \mathcal{B}$. Then a functor $\xi : T(\mathcal{X}) \rightarrow \mathcal{X}$ is constructed as follows: on objects, we put $\xi(X, a) := a_*(X)$ (=

⁴[Gray 1966] introduced the short hand “lali” for a left adjoint left inverse, like ξ .

$d_1(X \triangleright a)$), and on morphism we use the universal property of cocartesian arrows. More explicitly, ξ applied to the arrow \mathbf{a} in $T(\mathcal{X})$ over β displayed in (11) is the unique arrow $\xi(\mathbf{a})$ over β which makes the square in \mathcal{X}

$$\begin{array}{ccc}
 X & \xrightarrow{X \triangleright a} & a_*(X) \\
 \downarrow x & & \vdots \xi(\mathbf{a}) \\
 X' & \xrightarrow{X' \triangleright a'} & a'_*(X')
 \end{array} \tag{21}$$

commute.

The fact that $\xi \circ y_{\mathcal{X}} = id$ is left to the reader. The fact that ξ is indeed a left adjoint for $y_{\mathcal{X}}$ follows because it solves a universal problem; the unit of the adjunction at the object (X, a) is the arrow $(X, a) \rightarrow (a_*X, 1_B)$ in $T(\mathcal{X})$ given by the diagram

$$\begin{array}{ccc}
 X & & \\
 \searrow X \triangleright a & & \\
 & & a_*(X) \\
 : & & \\
 A & \xrightarrow{a} & B \\
 \searrow a & & \searrow 1_B \\
 & B & \xrightarrow{1_B} B
 \end{array}$$

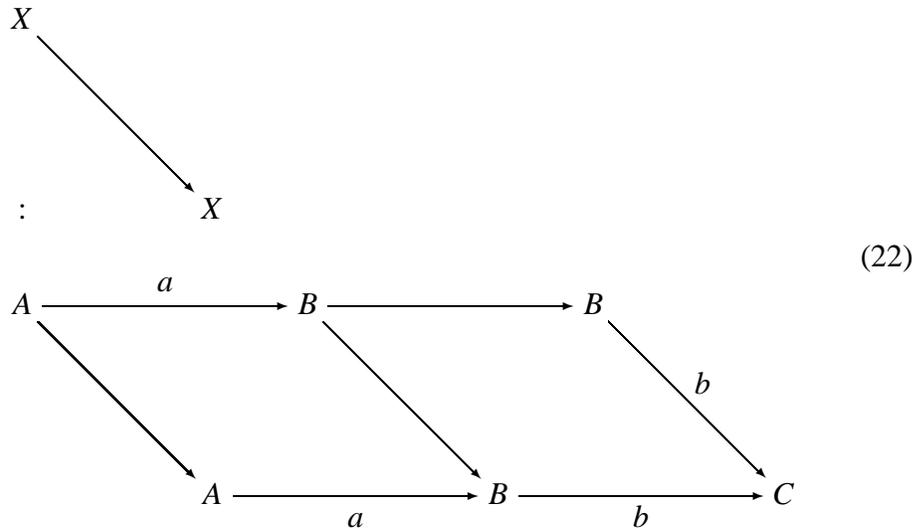
This proves the assertion 1) of the Theorem.

Assume next that ξ is associative, i.e. $\xi \circ T(\xi) = \xi \circ m$. The arrows picked out by the cleavage \triangleright derived from ξ are those that are of the form: ξ applied to an arrow in $T(\mathcal{X})$ of the form $((X, 1_A); a)$, as in (20); whereas ξ applied to a more general canonical cocartesian arrow in $T(\mathcal{X})$ of the form $((X, a); b)$, as exhibited in (12), is not apriori picked out by the cleavage \triangleright . However, we have, with \triangleright and the resulting $a_*(X)$ derived from ξ , the following

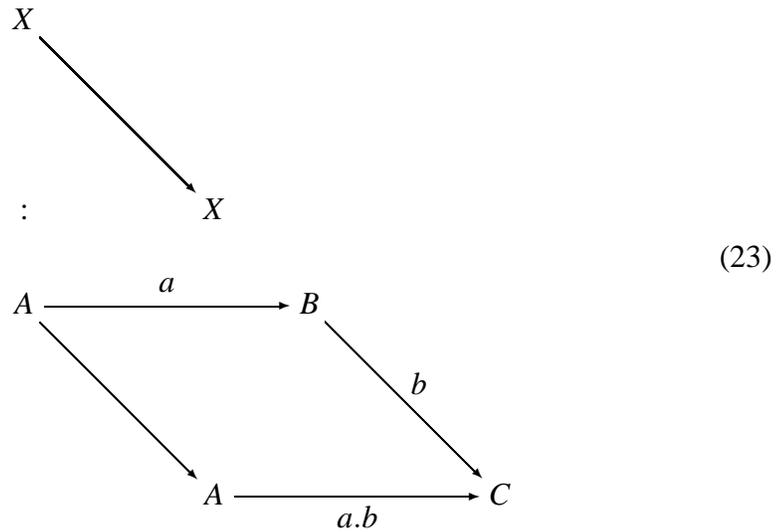
Lemma. 6.2 Assume that ξ is associative. Then

$$\xi((X, a); b) = a_*(X) \triangleright b.$$

Proof. Consider the following arrow over b in $T^2(\mathcal{X})$:



(unnamed arrows are identity arrows); applying $m_{\mathcal{X}}$ yields



i.e. $((X, a); b)$; whereas applying $T(\xi)$ to (22) yields

$$\begin{array}{ccc}
a_*(X) & & \\
\searrow & & \\
: & & a_*(X)
\end{array}
\tag{24}$$

$$\begin{array}{ccccc}
B & \xrightarrow{\quad} & B & & \\
& \searrow & & \searrow & \\
& & B & \xrightarrow{\quad} & C \\
& & & \text{\scriptsize } b & \\
& & & & \text{\scriptsize } b
\end{array}$$

(since $\xi(X, a) = a_*(X)$ and since the first square of (22) is an identity arrow in $T(\mathcal{X})$). By the strict associativity of ξ , the value of ξ on (23) and (24) is the same, and these values are $\xi((X, a); b)$ and $a_*(X) \triangleright b$, respectively. This proves the Lemma.

To prove the splitting condition (6), let an arrow $b : B \rightarrow C$ be given. Consider the composite in $T(\mathcal{X})$

$$(X, 1_A) \xrightarrow{((X, 1_A); a)} (X, a) \xrightarrow{((X, a); b)} (X, a.b). \tag{25}$$

One sees that applying ξ (using the Lemma for the second factor) gives the composite composite $(X \triangleright a) \cdot (a_*(X) \triangleright b)$. On the other hand, the composite (25) is $((X, 1_A); a.b)$, which by ξ gives $X \triangleright (a.b)$. This proves that the cleavage \triangleright produced by a strictly associative ξ is in fact a splitting.

Conversely, given a splitting \triangleright , then since \triangleright is in particular a normalized cleavage, it gives rise to a functor $\xi : T(\mathcal{X}) \rightarrow \mathcal{X}$, with $\xi \circ y_{\mathcal{X}}$ the identity functor on \mathcal{X} , as described above. It remains to prove that ξ satisfies the associative law $\xi \circ T(\xi) = \xi \circ m : T^2(\mathcal{X}) \rightarrow \mathcal{X}$. Consider an object (X, a, b) in $T^2(\mathcal{X})_B$, as

displayed in

$$\begin{array}{c}
 X \\
 : \\
 A \xrightarrow{a} B \xrightarrow{b} C.
 \end{array}$$

Then

$$\xi(T(\xi)(X, a, b)) = \xi(a_*(X), b) = b_*(a_*(X)).$$

On the other hand

$$\xi(m(X, a, b)) = \xi(X, a.b) = (a.b)_*(X),$$

and then (7) gives the associativity result, in so far as objects of $T^2(\mathcal{X})$ is concerned. Next, consider a morphism in $T^2(\mathcal{X})$ from (X, a, b) to (X', a', b') given by (x, β, γ) (with $\alpha = \pi(x)$), displayed as the full arrows in the three-dimensional diagram

$$\begin{array}{ccccccc}
 X & \xrightarrow{X \triangleright a} & a_*(X) & \xrightarrow{a_*(X) \triangleright b} & b_*(a_*(X)) & & \\
 \downarrow x & & \searrow \xi(x, \beta) & & \searrow \xi(\xi(x, \beta), \gamma) & & \\
 : & & X' & \xrightarrow{X' \triangleright a'} & a'_*(X') & \xrightarrow{a'_*(X') \triangleright b'} & b'_*(a'_*(X')) \\
 & & & & & & \\
 A & \xrightarrow{a} & B & \xrightarrow{b} & C & & : \\
 \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
 A' & \xrightarrow{a'} & B' & \xrightarrow{b'} & C' & &
 \end{array}$$

The slanted dotted arrows in the top layer are, by construction of the value of ξ on arrows in $T(\mathcal{X})$, the unique ones (over β and γ , respectively) which make the squares on the top commute.

So $\xi \circ T(\xi)$ applied to the given arrow in $T^2(\mathcal{X})$ is the rightmost slanted arrow on the top. On the other hand, $m_{\mathcal{X}}$ applied to the given morphism in $T^2(\mathcal{X})$ is given by the full arrows in

$$\begin{array}{ccc}
X & \xrightarrow{X \triangleright (a.b)} & (a.b)_*(X) \\
\downarrow x & & \searrow \xi(\xi(x, \beta), \gamma) \\
X' & \xrightarrow{X' \triangleright (a'.b')} & (a'.b')_*(X') \\
\vdots & & \vdots
\end{array}
\quad (26)$$

$$\begin{array}{ccc}
A & \xrightarrow{a.b} & C \\
\downarrow \alpha & & \searrow \gamma \\
A' & \xrightarrow{a'.b'} & C' \\
\vdots & & \vdots
\end{array}$$

so ξ applied to it is unique one over γ making the top square commute. But now by the splitting condition for \triangleright , equation (6), we conclude that the composite of \triangleright arrows in the previous diagram equals the \triangleright arrow in the present one, so by uniqueness of chosen cocartesian lifts of γ , we conclude that the two desired arrows in \mathcal{X} agree, proving $\xi \circ m = \xi \circ T(\xi)$.

It is clear that the two processes $\xi \leftrightarrow \triangleright$ are mutually inverse, and so the assertion 2) is proved.

Finally, consider two strict T -algebras (\mathcal{X}, ξ) and (\mathcal{X}', ξ') . We need to prove that a functor $F : \mathcal{X} \rightarrow \mathcal{X}'$ over \mathcal{B} is compatible with T -algebra structures ξ and ξ' iff it is compatible with the associated splittings \triangleright and \triangleright' . If F is compatible with the algebra structures, we get that F takes the cocartesian arrows $X \triangleright a$ in \mathcal{X} to the cocartesian arrow $F(X) \triangleright' a$ in \mathcal{X}' ; this follows by considering the canonical cocartesian arrow (20) and applying the two functors (assumed equal) $F \circ \xi$ and $\xi' \circ T(F)$ to it. Conversely, if $F(X \triangleright a) = F(X) \triangleright' a$, the functors $F \circ \xi$ and $\xi' \circ T(F)$ give equal value in \mathcal{X}' on the object $(X, a) \in T(\mathcal{X})$; for,

$$\begin{aligned}
F(\xi(X, a)) &= F(d_1(X \triangleright a)) = d_1(F(X \triangleright a)) = d_1(F(X) \triangleright' a) \\
&= \xi'(F(X), a) = \xi'(T(F)(X, a)).
\end{aligned}$$

The two functors in question then agree also on morphisms; this follows from the fact that they agree on objects, and from the fact that their values on morphisms is determined by universal properties. This proves assertion 3) and thus the Theorem.

The two last assertions of the Theorem immediately lead to

Corollary 6.3 *The category of split (op-)fibrations over \mathcal{B} , and strict splitting preserving functors, is monadic over Cat/\mathcal{B} by the KZ monad T .*

Also, the category of opfibrations with a cleavage is the category of pseudo-algebras for the monad T , with morphisms: functors over \mathcal{B} preserving the cleavages strictly; and with pseudo-morphisms: the functors over \mathcal{B} that preserve cocartesian arrows. We refrain from making these “pseudo-” notions explicit, but it should be mentioned that those natural isomorphisms that occur in the precise definition of the “pseudo-” notions for the case of KZ monads automatically satisfy the coherence conditions that usually must be required for such isomorphisms, because they solve universal problems.

7 Replacing cleavages with splittings

Every opfibration admits a normalized cleavage (granted the axiom of choice); but as remarked in the Example and Remark at the end of Section 3, it may not admit a splitting. On the other hand, one has ([Giraud] I.2):

Theorem 7.1 *Every opfibration $\pi : \mathcal{X} \rightarrow \mathcal{B}$ is equivalent (in the 2-category Cat/\mathcal{B}) to one with a splitting.*

Proof. Choose a normalized cleavage \triangleright for $\mathcal{X} \rightarrow \mathcal{B}$, and construct the corresponding left adjoint left inverse $\xi : T(\mathcal{X}) \rightarrow \mathcal{X}$ for $y_{\mathcal{X}}$. Then take the full image of ξ in \mathcal{X} . Recall that the full image $\mathcal{F}(\xi)$ of a functor $\xi : \mathcal{Y} \rightarrow \mathcal{X}$ has the same objects as \mathcal{Y} , and that the set of arrows $Y_1 \rightarrow Y_2$ in $\mathcal{F}(\xi)$ is the set of arrows $\xi(Y_1) \rightarrow \xi(Y_2)$ in \mathcal{X} . (If you want disjoint hom-sets in $\mathcal{F}(\xi)$, then put on some labels Y_i). There is an evident factorization of ξ :

$$\mathcal{Y} \xrightarrow{\xi_1} \mathcal{F}(\xi) \xrightarrow{\xi_2} \mathcal{X}$$

where ξ_1 is bijective on objects and ξ_2 is full and faithful. If ξ is surjective on objects, ξ_2 is an equivalence of categories. In our case, $\xi : T(\mathcal{X}) \rightarrow \mathcal{X}$ is surjective on objects, because $\xi \circ y_{\mathcal{X}}$ is the identity functor on \mathcal{X} . Also, it is easy

to see that the full image construction $\mathcal{F}(\xi)$ respects the “augmentations” to \mathcal{B} . So the proof is completed if we can provide $\mathcal{F}(\xi) \rightarrow \mathcal{B}$ with the structure of a split opfibration. But $T(\mathcal{X})$ carries a canonical splitting, which we can transfer to $\mathcal{F}(\xi)$ using ξ . If we denote the canonical splitting of $T(\mathcal{X})$ by \triangleright_0 , then we define a cleavage \triangleright_1 on $\mathcal{F}(\xi)$ by the formula

$$Y \triangleright_1 a \quad := \quad \xi_1(Y \xrightarrow{Y \triangleright_0 a} a_*(Y))$$

where $a_*(Y)$ denotes the codomain of $Y \triangleright_0 a$. We note that this arrow in $\mathcal{F}(\xi)$ has indeed the correct domain, namely $\xi_1(Y) = Y$; we also note that its codomain is again $a_*(Y)$. To prove that the cleavage \triangleright_1 is a splitting, we consider the equation $(Y \triangleright_0 a) \cdot (a_* Y \triangleright_0 b) = Y \triangleright_0 (a \cdot b)$, which holds, since \triangleright_0 is a splitting. We then get the desired equation for \triangleright_1 using that ξ_1 is a functor, and applying the definition of \triangleright_1 to each of the three terms of the equation.

8 Comparisons

Apparently, Chevalley was the one to formulate the notion of fibration in adjointness terms; [Gray 1966] (p. 56) uses the term “Chevalley Criterion” for the following (which I here state for opfibrations rather than fibrations):

a functor $\pi : \mathcal{X} \rightarrow \mathcal{B}$ is an opfibration iff the canonical functor $\bar{\pi} : \mathcal{X}^2 \rightarrow \pi \downarrow \mathcal{B}$ admits a left adjoint right inverse (lari) K .

Recall that the criterion considered presently is that $y_x : \mathcal{X} \rightarrow T(\mathcal{X})$ admits a (vertical) left adjoint *left inverse* (lali) ξ ; and recall also that $T(\mathcal{X})$ is $\pi \downarrow \mathcal{B}$ (seen as a category over \mathcal{B}). For a lari, it is the counit of the adjunction which carries the information (the unit being an identity); for a lali, it is the unit which carries the information. [Street 1974] analyzed (p.118-119), in abstract 2-categorical terms, that the data of K and ξ are equivalent. I shall here describe, in elementary terms, the passage from ξ to K , and describe the counit for the lari adjunction $K \dashv \bar{\pi}$. Recall the notation applied in the present article: for $(X, a) \in \pi \downarrow \mathcal{B}$, ξ returns the value $a_*(X)$, and the unit of the adjunction is essentially $X \triangleright a$. Out of this data, one constructs $K : \pi \downarrow \mathcal{B} \rightarrow \mathcal{X}^2$ by sending (X, a) to the arrow $X \triangleright a : X \rightarrow a_*(X)$. The counit of the lari adjunction $K \dashv \bar{\pi}$, instantiated at an object $x : X \rightarrow Y$ in \mathcal{X}^2

is an arrow in \mathcal{X}^2 , namely the commutative square

$$\begin{array}{ccc}
 X & \xrightarrow{X \triangleright \pi(x)} & \pi(x)_*(X) \\
 \downarrow 1_X & & \vdots \\
 X & \xrightarrow{x} & Y
 \end{array}$$

where the right hand vertical arrow comes from the universal property of the co-cartesian arrow on the top.

Street was probably the first to observe that opfibrations could be described as pseudo-algebras for a KZ monad; in fact, in [Street 1974] p. 118, he uses this description as his *definition* of the notion of opfibration, so therefore, no proof is given. Also, loc.cit. gives no proof of the fact that split opfibrations then are the strict algebras. So in this sense, Section 6 of the present article only supplements loc.cit. by providing elementary proofs of these facts.

References

- [1] F. Borceux (1994), Handbook of Categorical Algebra Vol. II, Cambridge University Press.
- [2] J. Giraud (1971), Cohomologie non abélienne, Grundlehren Band 179, Springer Verlag.
- [3] J. Gray (1966), Fibred and Cofibred Categories, in Proc. Conf. on Categorical Algebra, La Jolla, Springer Verlag.
- [4] P. Johnstone (2002) Sketches of an Elephant Vol.I, Oxford Logic Guides 43, Oxford University Press.
- [5] A. Kock (1973), Monads for which structures are adjoint to units (v.1), Aarhus Preprint Series 1972-73, 35.
- [6] A. Kock (1995), Monads for which structures are adjoint to units (v.3), Journ. Pure Appl. Algebra 104, 41-59.

- [7] R. Street (1974), Fibrations and Yoneda's Lemma in a 2-category, in Sydney Category Seminar 1972/73, Springer Lecture Notes 420.

Diagrams were made with Paul Taylor's package.

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