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Cubical Version of Combinatorial Differential Forms

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Abstract The theory of combinatorial differential forms is usually presented in simplicial terms. We present here a cubical version; it depends on the possibility of forming affine combinations of mutual neighbour points in a manifold, in the context of synthetic differential geometry.

Keywords Cubical combinatorial form · Synthetic differential geometry

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1 Introduction

The motivation for the present investigation of a "cubical" formulation of combinatorial differential forms (as in [7] I.18, [1, 10]) was a desire to generalize it to a combinatorial theory of higher connections in higher groupoids.

For the 1-dimensional case, (connections in groupoids), such combinatorial notion exists [8]: a 1-connection in a groupoid over a manifold M is a morphism of a certain reflexive graph $M_{(1)} \rightrightarrows M$ into the (underlying graph of) the groupoid. (The domain graph here is the graph of 1-dimensional infinitesimal simplices in M, or, equivalently, the first neighbourhood of the diagonal of M.) If the groupoid is a "constant" groupoid $M \times G$ for a group G, such connection is equivalent to a combinatorial G-valued 1-form on M.

Now $M_{(1)} \rightrightarrows M$ is the 1-skeleton of that simplicial complex which carries the theory of combinatorial differential forms. But it turns out also to be the 1-skeleton

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of a *cubical* complex $M_{[\bullet]}$, as will be explained. There already exists a notion of higher "cubical groupoid" [2]; the 2-dimensional case is essentially the notion of "special double groupoid" or "edge symmetric double groupoid" of [3]. This opens up for the possibility of considering higher connections as morphisms of cubical complexes, from $M_{[\bullet]}$ into the underlying cubical complex of a cubical groupoid. This turns out to be a natural frame for discussing things like curvature, Bianchi Identity, and holonomy, of higher connections, generalizing the 1-dimensional combinatorial theory in [8] and [11]. (This is the subject of a forthcoming paper, a preliminary account of which is in [14]; see also Section 8 for some sketchy indications.)

Any abelian group R defines a k-dimensional "constant" cubical groupoid on M, and a connection in this cubical groupoid is tantamount to a cubical-combinatorial differential k-form, – a notion to be defined and developed in the present note. Under this correspondence, curvature corresponds to exterior derivative (=coboundary), and holonomy corresponds to integration of differential forms.

This note is part of the theory that I presented at the Carvoeiro meeting 2007 (see also [14]); there, however, the emphasis was on the general theory of higher connections, leaving the theory of cubical differential forms implicit as a special case. Here, we do the special case with a little more patience, since differential forms are anyway more basic, and of independent interest.

It should also be said that the cubical version of combinatorial differential forms makes the integration theory more transparent than the simplicial version. The reason is essentially that any cube naturally can be subdivided into smaller cubes. This, we develop in Sections 5 and 6 below.

The context of the theory here (from Section 4 on) is that of Synthetic Differential Geometry (SDG), cf. [7] and references therein.

2 The Complex of Singular Cubes

In the present section, we place ourselves in the context of smooth manifolds. So all maps mentioned are smooth. The content is probably not new; it is formulated in completely classical terms, and we shall be brief. For a manifold M, we consider "singular k-cubes in M"; by this is usually meant maps $I^k \to M$, where $I = \{x \in R \mid 0 \leq x \leq 1\}$; however, to simplify things, we do not want to consider a partial order $\leq$ on the number line R (hence there is no such ting as the unit interval I); we prefer to define singular cubes as maps $R^k \to M$. The set of these is denoted $S_{[k]}(M)$.

As k ranges, the sets $S_{[k]}(M)$ form a cubical complex, which we denote $S_{[\bullet]}(M)$. It has face- and degeneracy maps, induced by certain affine maps $R^{k\pm 1} \to R^k$ (partly recalled below); it has a symmetry structure: the symmetric group in k letters acts on $S_{[k]}(M)$, by an action induced by permutation of the coordinates of R^k; it also has a "reversion" structure [5], which we shall not need here (although it is important in [14]). Finally, it has also a subdivision structure. This structure is likewise induced by certain affine maps $R^k \to R^k$, and this is crucial for the present theory.

The face maps $\partial_i^\alpha : S_{[k]}(M) \to S_{[k-1]}(M)$ ($\alpha = 0$ or 1, $i = 1, \ldots, k$) are induced by the maps $\delta_i^\alpha : R^{k-1} \to R^k$ given by

$$\delta_i^\alpha(t_1, \ldots, t_{k-1}) = (t_1, \ldots, \alpha, \ldots, t_{k-1})$$

with α inserted as ith coordinate. Then for a singular $\gamma : R^k \to M$, $\partial_i^\alpha(\gamma) := \gamma \circ \delta_i^\alpha$. The degeneracy operators are similarly precomposition by the projection maps $R^k \to R^{k-1}$; we don't need to be more specific here.

We do, however, need to be specific about the notion of subdivision. First, for any $a, b \in R$, we have an affine map $R \to R$, denoted $[a, b] : R \to R$; it is the unique affine map sending 0 to a and 1 to b, and is given by $t \mapsto (1-t) \cdot a + t \cdot b$. Precomposition with it, $\gamma \mapsto \gamma \circ [a, b]$ defines an operator $S_{[1]}(M) \to S_{[1]}(M)$ which we denote $\mid [a, b]$, and we let it operate on the right, thus

$$\gamma \mid [a, b] := \gamma \circ [a, b].$$

Heuristically, it represents the restriction of γ to the interval $[a, b]$. Note that $\gamma = \gamma \mid [0, 1]$, since $[0, 1] : R \to R$ is the identity map.

More generally, for $a_i, b_i \in R$ for $i = 1, \dots, k$, we have the map

$$[a_1, b_1] \times \dots \times [a_k, b_k] : R^k \to R^k.$$

It induces by precomposition an operator $S_{[k]}(M) \to S_{[k]}(M)$, which we similarly denote $\gamma \mapsto \gamma \mid [a_1, b_1] \times \dots \times [a_k, b_k]$.

Given an index $i = 1, \dots, k$, and $a, b \in R$. We use the abbreviated notation $[a, b]_i$ for the map $[0, 1] \times \dots \times [a, b] \times \dots \times [0, 1]$ with the $[a, b]$ appearing in the ith position; the corresponding operator is denoted $\gamma \mapsto \gamma \mid_i [a, b]$. Given a, b and $c \in R$, and an index $i = 1, \dots, k$, we say that the ordered pair

$$\gamma \mid_i [a, b], \quad \gamma \mid_i [b, c]$$

forms a subdivision of $\gamma \mid_i [a, c]$ *in the ith direction.*

There are compatibilities between the subdivision relation and the face maps; we shall record some of these relations. First, let us note that the maps $\delta_i^\alpha : R^{k-1} \to R^k$ considered above for $\alpha = 0$ or $= 1$, may be similarly defined for any $\alpha = a \in R$; namely

$$\delta_i^a(t_1, \dots, t_{k-1}) = (t_1, \dots, t_{i-1}, a, t_i, \dots, t_{k-1}).$$

The corresponding $S_{[k]}(M) \to S_{[k-1]}(M)$ we denote of course ∂_i^a. It is easy to see that we have $[a, b]_i \circ \delta_i^0 = \delta_i^a$, and similarly $[a, b]_i \circ \delta_i^1 = \delta_i^b$, from which follows, for any $\gamma \in S_{[k]}(M)$, that

$$\partial_i^0(\gamma \mid_i [a, b]) = \partial_i^a(\gamma) \text{ and } \partial_i^1(\gamma \mid_i [a, b]) = \partial_i^b(\gamma). \tag{1}$$

Also, for $\alpha = 0, 1$ (in fact for every $\alpha = a \in R$) $[a, b]_i \circ \delta_j^\alpha = \delta_j^\alpha \circ [a, b]_i$ if $i < j$ and $= \delta_j^\alpha \circ [a, b]_{i-1}$ if $i > j$, and from this follows, for any $\gamma \in S_{[k]}(M)$, that

$$\partial_j^\alpha(\gamma \mid_i [a, b]) = \left(\partial_j^\alpha(\gamma)\right) \mid_i [a, b] \text{ for } i < j \tag{2}$$

and $= (\partial_j^\alpha(\gamma)) \mid_{i-1} [a, b]$ for $i > j$.

Recall that an *affine combination* in a vector space is a linear combination where the sum of the coefficients is 1. An affine space is a set E where one may form affine combinations, and where these combinations satisfy the same equations as those that are valid for affine combinations in vector spaces. An affine map is a map preserving affine combinations. The vector space R^n is a free affine space on $n + 1$ generators. More concretely, given a $n + 1$-tuple of points $(x_0, x_1, \dots, x_n)$ in an affine space E, there is a unique affine map $R^n \to E$, which we denote $[x_0, x_1, \ldots, x_n]$, with $0 \mapsto x_0$ and $e_i \mapsto x_i$ for $i > 0$, where e_j is the jth canonical basis vector $e_j \in R^n$.

This map $[x_0, x_1, \ldots, x_n]$ is given by

$$(t_1, \ldots, t_n) \mapsto \left(1 - \sum t_i\right) x_0 + t_1 x_1 + \ldots + t_n x_n. \tag{3}$$

An affine map between vector spaces is of the form: a constant plus a linear map. This allows us to have a matrix calculus for affine maps between the coordinate vector spaces R^n. Recall that a linear map $f : R^n \to R^m$ is given by an $m \times n$ matrix, and that composition of maps corresponds to matrix multiplication. The jth column $a_j \in R^m$ of such matrix A is $f(e_j)$ $(e_j \in R^n)$.

An affine map $R^n \to R^m$ may be given by an $m \times (1 \times n)$ matrix, where the first column $a_0 \in R^m$ denotes the constant, and the remaining $m \times n$ matrix A is the $m \times n$ matrix of the linear map. We display this "augmented" matrix in the form $[a_0 \mid A]$ or $[a_0 \mid a_1, \ldots, a_n]$ where as before a_j $(j = 1, \ldots, n)$ is the jth column of A.

With this notation, composition of affine maps corresponds to "semi-direct matrix multiplication":

$$[a_0 \mid A] \cdot [b_0 \mid B] = [a_0 + A \cdot b_0 \mid A \cdot B].$$

For $E = R^m$, the affine map $[x_0, \ldots, x_n] : R^n \to R^m$ considered above has as augmented matrix the matrix $[x_0 \mid x_1 - x_0, \ldots, x_n - x_0]$, and conversely, the augmented matrix $[x_0 \mid a_1, \ldots, a_n]$ defines the affine map $[x_0, x_0 + a_1, \ldots, x_0 + a_n]$.

Let us also give the augmented matrices for the affine maps δ_i^α that were used for defining the cubical face maps ∂_i^α, $\alpha = 0$ or 1, $i = 1, \ldots, k$:

$$\delta_i^0 = [0 \mid e_1, \ldots, \widehat{i}, \ldots, e_k] \tag{4}$$

(the e_js here are the canonical basis vectors of R^k), and

$$\delta_i^1 = [e_i \mid e_1, \ldots, \widehat{i}, \ldots, e_k]. \tag{5}$$

With the matrix calculus for augmented matrices, we can calculate the cubical faces of a singular k-cube in R^n of the form $[x_0, x_1, \ldots, x_k]$; We have

$$\partial_i^0([x_0, x_1, \ldots, x_k]) = [x_0, x_1, \ldots \widehat{x_i}, \ldots, x_k] \tag{6}$$

and

$$\partial_i^1([x_0, x_1, \ldots, x_k]) = [x_i, x_1 - x_0 + x_i, \ldots, \widehat{i}, \ldots, x_k - x_0 + x_i]. \tag{7}$$

(Note that the entries like $x_k - x_0 + x_i$ are affine combinations , so they also make sense for points in an affine space E; $x_k - x_0 + x_i$ is the fourth vertex (opposite x_0) of a parallelogram whose three other vertices are x_0, x_i and x_k. If the x_is are mutual neighbours in a manifold, as elaborated in Section 4 below, it also makes sense there.)

Among the affine maps $[x_0, x_1, \ldots, x_k]$ from R^k to R^k are the "axis-parallel rectangular boxes", for short: *rectangles*; they are those where $x_i - x_0$ ("the ith side") is of the form $t_i \cdot e_i$ where e_i is the ith canonical basis vector. In matrix terms, these are matrices of the form $[x_0 \mid T]$ where T is a diagonal matrix. Let us spell out a subdivision for rectangles in matrix terms:

$$\left[\begin{array}{c|cccc} x_{01} & t_1 & & & \\ \vdots & & \ddots & & \\ x_{0i} & & & t_i + s_i & \\ \vdots & & & & \ddots \\ x_{0k} & & & & & t_k \end{array}\right] \tag{8}$$

is subdivided in the ith direction into

$$\left[\begin{array}{c|cccc} x_{01} & t_1 & & & \\ \vdots & & \ddots & & \\ x_{0i} & & & t_i & \\ \vdots & & & & \ddots \\ x_{0k} & & & & & t_k \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|cccc} x_{01} & t_1 & & & \\ \vdots & & \ddots & & \\ x_{0i} + t_i & & & s_i & \\ \vdots & & & & \ddots \\ x_{0k} & & & & & t_k \end{array}\right]$$

We invite the reader to spell out subdivisions of $[x_0, x_1, \ldots, x_k]$ explicitly, and in particular to prove that $[x_0, x_1, \ldots, x_0, \ldots, x_k]$ (with x_0 appearing again in the ith position) subdivides in the ith direction into two copies of itself.

3 The Chain Complexes of Singular Cubes

Out of the cubical complex $S_{[\bullet]}(M)$, we can manufacture a chain complex $C_\bullet(M)$ in the standard way. We let $C_k(M)$ be the free abelian group generated by $S_{[k]}(M)$. The boundary operator $\partial : C_k(M) \to C_{k-1}(M)$ is defined on the generators $\gamma \in S_{[k]}(M)$ by the standard formula (see e.g. [6] 8.3) with $2k$ terms

$$\partial(\gamma) := \sum_{i=1}^{k} (-1)^i \big(\partial_i^0(\gamma) - \partial_i^1(\gamma)\big). \tag{9}$$

We let $N_k(M) \subseteq C_k(M)$ be the subgroup generated by

$$\gamma - \gamma' - \gamma''$$

for all γ which are subdivided in some direction into γ' and γ'.

Proposition 1 *The boundary operator $\partial : C_k(M) \to C_{k-1}(M)$ maps N_k into N_{k-1}.*

Proof Assume γ is subdivided in the ith direction into γ' and γ''. By (2) we have that, for $j \neq i$, $\partial_j^\alpha(\gamma)$ is subdivided, (in the ith direction, or in the $i-1$th direction, according to whether $j > i$ or $j < i$) into $\partial_j^\alpha(\gamma')$ and $\partial_j^\alpha(\gamma'')$; the difference of these terms is in N_{k-1}. In $\partial(\gamma - \gamma' - \gamma'')$ only remain the six ∂_i^α-terms. Omitting i from notation, these sum of these six terms are (plus or minus)

$$[\partial^0(\gamma) - \partial^0(\gamma') - \partial^0(\gamma'')] - [\partial^1(\gamma) - \partial^1(\gamma') - \partial^1(\gamma'')].$$

The two first terms in the left hand square bracket cancel by (1), and the two outer terms in the last square bracket cancel for the same reason . So we are left with $\partial^1(\gamma') - \partial^0(\gamma'')$. This is 0, likewise by (1). This proves the Proposition. □

We have the cochain complex of R-valued cochains on the cubical complex just described. A k-cochain on M is thus a map $\Phi : S_{[k]}(M) \to R$, or equivalently an additive map $\Phi : C_k(M) \to R$. Such cochains behave contravariantly, i.e. given a map $f : M' \to M$ between manifolds and a cochain Φ on M, we get a cochain $F^*(\Phi)$ on M'.

Also, the (cubical) boundary operator $\partial : C_{k+1}(M) \to C_k(M)$ gives rise to a coboundary operator d from k-cochains to $k+1$-cochains.

A k-cochain Φ is said to satisfy the *subdivision law* if

$$\Phi(\gamma) = \Phi(\gamma') + \Phi(\gamma'')$$

whenever a singular cube γ subdivides, in some direction, into γ' and γ''. This is equivalent to saying that $\Phi : C_k(M) \to R$ kills $N_k(M)$.

A k-cochain Φ is called *alternating* if $\Phi(\gamma \circ \sigma) = \text{sign}(\sigma) \cdot \Phi(\gamma)$, for any map $\sigma : R^k \to R^k$ given by some permutation σ of the coordinates.

Definition 1 Consider a k-cochain, i.e. a map $\Phi : S_{[k]}(M) \to R$. It is called an *observable* (of dimension k) if it satisfies the subdivision law and is alternating.

(The term "observable", I picked up from Meloni and Rogora [17], who considered such functionals, for similar reasons as ours. I don't know the motivation for the terminology. Similar notions appear in Félix and Lavendhomme's [4], reproduced also in [16] 4.5.3.)

Proposition 2 *If Φ is an observable, then so is $d\Phi$.*

Proof Since $N_\bullet(M)$ is stable under the boundary operator, it follows that if Φ satisfies the subdivision law, then so does $d\Phi$. Next, for the alternating property: It suffices to consider those permutations σ_i which interchange ith and $i+1$st coordinate. We must prove that $\Phi(\partial(\gamma \circ \sigma_i)) = -\Phi(\partial(\gamma))$. Writing $\partial_j^\alpha(\gamma)$ in terms of its definition by the affine maps $\delta_j^\alpha : R^k \to R^{k+1}$,

$$\partial(\gamma \circ \sigma) = \sum_{j=1}^{k+1} (-1)^j \big(\gamma \circ \sigma_i \circ \delta_j^0 - \gamma \circ \sigma_i \circ \delta_j^1\big).$$

Now we need some relations between the σ_i and δ_j^α. They can be found in formula [5] (29) (middle line). Let us elaborate on the case $i = 1$, and leave the remaining cases to the reader. For $j \geq 3$, $\sigma_1 \circ \delta_j^\alpha = \delta_j^\alpha \circ \sigma_1$, so when applying Φ, we get the required sign change. There remains the terms $j = 2$ and $j = 1$. Here, the sign change occurs already at the level of the chain complex: The $j = 2$-term of the chain $\partial(\gamma \circ \sigma_1)$ is

$$(-1)^2 \big(\gamma \circ \sigma_1 \circ \delta_2^0 - \gamma \circ \sigma_1 \circ \delta_2^1\big);$$

now, by loc.cit. $\sigma_1 \circ \delta_2^\alpha = \delta_1^\alpha$, so the $j = 2$ term in $\partial(\gamma \circ \sigma_1)$ equals minus the $j = 1$-term in $\partial(\gamma)$. Similarly, the $j = 1$-term in $\partial(\gamma \circ \sigma_1)$ equals minus the $j = 2$-term in $\partial(\gamma)$, because $\sigma_1 \circ \delta_1^\alpha = \delta_2^\alpha$ by loc.cit.

This proves the Proposition. □

We thus have a cochain complex of observables; but we don't give it a name, since we shall see that this is isomorphic to the cochain complex of (cubical) differential forms, see Section 6.

4 Combinatorial Cubical Forms

We now place ourselves in the context of SDG, e.g. in a well adapted (topos) model which contains the category of smooth manifolds in a good way as a full subcategory, but also contains sufficiently many infinitesimal objects, cf. [7] and the references therein. In this context, any manifold M acquires a reflexive and symmetric (not transitive) *neighbour* relation,[1] denoted $\sim$. For $M = R$, $x \sim y$ iff $(x - y) \in D$, where $D \subseteq R$ is the set of elements $d \in R$ with $d^2 = 0$. More generally, for vectors $x, y \in R^n$, $x \sim y$ if $(x - y) \in D(n)$, where $D(n)$ is the set of vectors $(d_1, \dots, d_n)$ such that $d_i \cdot d_j = 0$ for all $i, j = 1, \dots n$. A basic axiom scheme ("KL axiom", see e.g. [16] 2.1.3, satisfied in the well adapted models) implies that any map $D(n) \to R$ extends uniquely to an affine map $R^n \to R$. In particular, a map $D(n) \to R$ which takes 0 to 0 extends uniquely to a linear map $R^n \to R$. In this sense, $D(n)$ is a "linear-map classifier". It follows that $D(n)^k$ similarly is a "multilinear map classifier", i.e. a map $D(n)^k \to R$ which takes value 0 if one of the k input vectors is 0, extends uniquely to a k-linear map $(R^n)^k \to R$.

The following infinitesimal object $\widetilde{D}(k, n) \subseteq D(n)^k$ was introduced in [7] I.16: it consists of k-tuples of vectors d_i in $D(n)$ which are furthermore mutual neighbours, $d_i - d_j \in D(n)$. We are here in particular interested in $\widetilde{D}(k, k)$. The crucial property of it, cf. loc.cit., (and likewise a consequence of the KL axiom scheme) is that $\widetilde{D}(k, k)$ is a k-linear *alternating* map classifier, i.e. a map $\widetilde{D}(k, k) \to R$ which takes value 0 if one of the k input vectors is 0, extends uniquely to a k-linear alternating map $(R^k)^k \to R$, and thus is a constant times the determinant formation of $k \times k$ matrices. (A remarkable property of matrices in $\widetilde{D}(k, k)$ is that the $k!$ terms in the usual sum formula for determinants are all equal; see [12].)

By an *infinitesimal k-simplex* in a manifold M we understand, like in [7], a $k+1$-tuple $(x_0, x_1, \dots, x_k)$ of mutual neighbour points in M. Note that if $(d_1, \dots, d_k) \in \widetilde{D}(k, n)$, then the $k+1$ tuple $(0, d_1, \dots, d_k)$ is an infinitesimal k-simplex in R^n; and any infinitesimal k-simplex in R^n with first entry $= 0$ is of this form.

It was proved in [9] (see also [10]) that if $(x_0, x_1, \dots, x_k)$ is an infinitesimal k-simplex in a manifold M, then affine combinations $\sum_0^k t_i \cdot x_i$ may be formed, using a coordinate chart, but independent of the chart chosen; furthermore, any two of these affine combinations are neighbours. Also equations between affine combinations of these affine combinations behave as if they were formed in a genuine affine space. And any map $M \to N$ preserves the neighbour relation, and preserves the formation of affine combinations of mutual neighbour points.

[1] A way of extending the notion of "neighbour relation" to a larger class of spaces than finite dimensional manifolds has been begun in [13]; but one should not expect such theory to be as simple as in the finite dimensional case.

The possibility of forming affine combinations of mutual neighbour points implies that the formula (3) describing the map $[x_0, \ldots, x_k]$, in case M is an affine space, makes sense even if M is just a manifold, provided $(x_0, \ldots, x_k)$ is an infinitesimal simplex. Thus such infinitesimal simplex in M defines a singular k-cube $[x_0, \ldots, x_k] : R^k \to M$ in M.

Definition 2 An *infinitesimal k-dimensional parallelepipedum* in M is a singular k-cube $R^k \to M$ which arises in this way from an infinitesimal k-simplex.

Note that there is a bijective correspondence between infinitesimal k-simplices and infinitesimal k-dimensional parallelepipeda in M. (Warning: the "infinitesimal singular rectangles" of [7], or the "marked microcubes" of [16], are in [15], and in [18] IV.1, called "infinitesimal k-cubes"; these things are not infinitesimal parallelepipeda in the sense of the present Definition. A comparison is made in Section 7.)

The set of infinitesimal k-dimensional parallelepipeda in M we shall denote $M_{[k]}$. The infinitesimal parallelepipeda in M, as k ranges, form a subcomplex of the cubical complex $S_{[\bullet]}(M)$, stable under the symmetry operations. Also, a subdivision of an infinitesimal parallelepipedum consists of two infinitesimal parallelepipeda. – We denote the cubical complex of infinitesimal parallelepipeda by $M_{[\bullet]}$.

Remark The set of infinitesimal k-simplices, as k ranges, naturally organize themselves into a *simplicial* complex $M_{<\bullet>}$, out of which grows the *simplicial* theory of combinatorial differential forms, [1, 7, 10].) So we have a somewhat surprising phenomenon that we have a cubical complex $M_{[\bullet]}$ and a simplicial complex $M_{<\bullet>}$ with isomorphic underlying graded sets.

A map $\omega : M_{[k]} \to R$ may be called an *R-valued infinitesimal cubical k-cochain* on M. As k ranges, they form a cochain complex with the coboundary formula for cochains derived from the boundary formula (9) for the chain complex associated to a cubical complex. We shall not give a name to this complex, since we are more interested in a certain subset (actually, a subcomplex), consisting of the *cubical-combinatorial differential forms*, as defined below. We first need a "degeneracy notion": by a *degenerate* infinitesimal k-dimensional parallelepipedum, we shall understand one of the form $[x_0, x_1, \ldots, x_k]$ with $x_i = x_0$ for some $i > 0$.

Definition 3 A *cubical-combinatorial differential k-form* (briefly, a cubical k-form) on M is a map (cochain) $M_{[k]} \to R$ which takes degenerate infinitesimal parallelepipeda to 0.

When we in the following say "k-form" without further decoration, we mean "cubical-combinatorial differential k-form". Also, d denotes the coboundary operator in the cochain complex associated to the cubical complex $M_{[\bullet]}$. (It is denoted d_c in [14], "c" for "cubical".)

If ω is a k-form on M, and $(x_0, \ldots, x_k)$ is a $k+1$-tuple of mutual neighbours points (i.e. an infinitesimal k-simplex) in M, we sometimes write $\omega(x_0, \ldots, x_k)$ rather than $\omega([x_0, \ldots, x_k])$.

Proposition 3 *If ω is a k-form, then $d\omega$ is a $k+1$-form.*

Proof This is a matter of checking that $d\omega$ vanishes on degenerate $k+1$-dimensional parallelepipeda. Let γ be a degenerate $k+1$-dimensional parallelepipedum,

$$\gamma = [x_0, x_1, x_2, \ldots, x_{k+1}]$$

with $x_i = x_0$ for some $i > 0$; for simplicity, let us assume $x_1 = x_0$. Now all faces of γ, except possibly the first, are themselves degenerate, so ω vanishes. The two faces $\partial_1^0(\gamma)$ and $\partial_1^1(\gamma)$ are equal, by (6) and (7), and so cancel. □

Remark In a forthcoming book, we prove that the cochain complex of differential k-forms in this sense is isomorphic to the deRham complex of M; the cubical coboundary d_c equals exterior derivative. The analogous assertion for the *simplicial* version of combinatorial differential forms was proved in [10], but requires modification by scalar factors of the type $(k+1)!$; the cubical case is actually easier.

We note that the various complexes described are functorial w.r.to maps $f : M \to N$ between manifolds. This follows because any such f preserves the neighbour relation, and also preserves the formation of affine combinations of mutual neighbour points. Thus, for θ a differential k-form on N, we get a differential k-form $f^*(\theta)$ on M by $f^*(\theta)(x_0, \ldots, x_k) := \theta(f(x_0), \ldots, f(x_k))$, where the x_is are mutual neighbours in M.

If θ is a k-form on N and $g : N \to R$ is a function, we get a new k-form $g \cdot \theta$ on N by putting

$$(g \cdot \theta)(x_0, \ldots, x_k) := g(x_0) \cdot \theta(x_0, \ldots, x_k).$$

(The privileged role of x_0 in the formula is spurious; it can be proved that choosing any of the other x_is will produce the same $g \cdot \theta$.) The multiplication thus defined is functorial: if $f : M \to N$ is a map, we have

$$f^*(g \cdot \theta) = f^*(g) \cdot f^*(\theta), \tag{10}$$

where $f^*(g)$ denotes $g \circ f$.

We need to know the structure of k-forms on R^k. There is a privileged one, called the *volume form*, and denoted Vol or $dx_1 \wedge \ldots \wedge dx_k$, given by

$$\mathrm{Vol}(x_0, x_1, \ldots, x_k) := \det(x_1 - x_0, \ldots, x_k - x_0).$$

It can be proved that any k-form ω on R^k is of the form $\widetilde{\omega} \cdot \mathrm{Vol}$ for a unique function $\widetilde{\omega} : R^k \to R$. (For, this is well known for differential forms in the classical sense; the bijection between classical forms and (simplicial) combinatorial forms was established in [7], Corollary I.18.4. Finally, there is an evident bijection between simplicial-combinatorial forms, and cubical-combinatorial forms; it is just the coboundaries that are different.)

Let $\alpha : R^k \to R^k$ be a map. Then $\alpha^*(\mathrm{Vol})$ is a k-form on R^k, and therefore it may be written $\alpha^*(\mathrm{Vol}) = J\alpha \cdot \mathrm{Vol}$ for a unique function $J\alpha : R^k \to R$ ("the Jacobi determinant of α"). From (10) then follows that

$$\alpha^*(f \cdot \mathrm{Vol}) = (f \circ \alpha) \cdot J\alpha \cdot \mathrm{Vol}\,. \tag{11}$$

We shall not prove in general that $J\alpha$ comes about from determinants of the Jacobi matrices, but shall only do it for the case of an *affine* map $\alpha : R^k \to R^k$, which is what we need here. Let α be $a + A$, with a constant $\in R^k$, and with $A : R^k \to R^k$ linear. Then for an infinitesimal k-simplex $(x_0, x_1, \ldots, x_k)$ in R^k,

$$\begin{aligned}(\alpha^*(\mathrm{Vol}))(x_0, x_1, \ldots, x_k) &= \mathrm{Vol}(\alpha(x_0), \alpha(x_1), \ldots, \alpha(x_k)) \\ &= \mathrm{Vol}(a + A(x_0), a + A(x_1), \ldots, a + A(x_k)) \\ &= \mathrm{Vol}(A(x_0), A(x_1), \ldots, A(x_k)) \\ &= \det(A(x_1) - A(x_0), \ldots, A(x_k) - A(x_0)) \\ &= \det(A(x_1 - x_0), \ldots, A(x_k - x_0))\end{aligned}$$

since A is linear

$$= \det(A) \cdot \det(x_1 - x_0, \ldots, x_k - x_0)$$

by the product rule for determinants. But the expression on the right is $\det(A) \cdot \mathrm{Vol}(x_0, x_1, \ldots, x_k)$. This proves that for an affine $\alpha : R^k \to R^k$, $\alpha^*(\mathrm{Vol}) = \det(A) \cdot \mathrm{Vol}$, where A is the linear part of α. Equivalently, for an affine α, $J\alpha$ is the function $R^k \to R$ with constant value $\det(A)$.

(The case of a not necessarily affine map $\alpha : R^k \to R^k$ is an easy consequence, since on the set of neighbours of x_0, like the x_is, α in any case agrees with an affine map, whose linear part is represented by the Jacobi matrix of α at x_0.)

5 Integration of Differential Forms

If ω is a (cubical-combinatorial) k-form on a manifold M, and $\gamma : R^k \to M$ a singular cube in M, we would like to define the *integral* of ω *along* γ, to be denoted $\int_\gamma \omega$. It should be functorial w.r.to maps $f : M \to N$ between manifolds, i.e. if θ is a k-form on N, and γ a singular k-cube on M, the integral should satisfy

$$\int_\gamma f^*(\theta) = \int_{f \circ \gamma} \theta.$$

Therefore, the essence resides in defining integrals $\int_{\mathrm{id}} \omega$, where id is the "generic k-cube", i.e. the identity map $R^k \to R^k$, and ω is a k-form on R^k. Recall that the identity map $R^k \to R^k$ may be described as the map $[0, 1] \times \ldots \times [0, 1]$, so $\int_{\mathrm{id}} \omega$ is also denoted $\int_{[0,1]\times\ldots\times[0,1]} \omega$.

To define these integrals, recall that a k-form ω on R^k may be written $\widetilde{\omega} \cdot \mathrm{Vol}$ for a unique function $\widetilde{\omega} : R^k \to R$. Therefore, we can define

$$\int_{[0,1]\times\ldots\times[0,1]} \omega := \int_0^1 \ldots \int_0^1 \widetilde{\omega}(t_1, \ldots, t_k)\, dt_k \ldots dt_1.$$

The right hand side is an ordinary iterated integral, and as such is something that can be dealt with in the synthetic context, without reference to "approximation by Riemann sums". Let us be more explicit. For the case $k = 1$, and for any $a, b \in R$, we may define

$$\int_a^b f(t)\,dt := F(b) - F(a),$$

where F is any anti-derivative of f, i.e. F is a function $F : R \to R$ with $F' = f$. This definition does not depend on whether $a \leq b$, in fact, we have not assumed any order relation $\leq$ on R. The chain rule $(f \circ \alpha)' = (f' \circ \alpha) \cdot \alpha'$ immediately leads to the substitution rule for one-dimensional integrals,

$$\int_a^b f(\alpha(t)) \cdot \alpha'(t)\,dt = \int_{\alpha(a)}^{\alpha(b)} f(t)\,dt \tag{12}$$

where α and f are functions $R \to R$.

If $f : R^k \to R$ is a function, and $a_1, \ldots, a_k, b_1, \ldots, b_k \in R$, we define the iterated integral in the expected way, by iteration of one-dimensional integrals.

All this can be copied out of the Calculus Books.

One reason why iterated integrals in general are not sufficient for calculus is the fact that they are only defined over *rectangular boxes* (and a few other simple kind of regions); therefore a theory of substitution in such integrals cannot be well formulated, say substitution along an arbitrary map $R^k \to R^k$; such a map may destroy rectangular shape.

However, the success of the substitution rule (12) for one-variable integrals does leave some trace on the theory of iterated integrals, namely substitution along maps $\alpha : R^k \to R^k$ of the form $\alpha = \alpha_1 \times \ldots \times \alpha_k : R^k \to R^k$, where each α_i is a map $R \to R$. Then the one-variable rule generalizes into

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\ldots\int_{a_k}^{b_k} f(\alpha(\underline{x})) \cdot \alpha'(x_1) \cdot \ldots \cdot \alpha'(x_k)\,dx_k \ldots dx_1$$
$$= \int_{\alpha_1(a_1)}^{\alpha_1(b_1)}\ldots\int_{\alpha_k(a_k)}^{\alpha_k(b_k)} f(x_1, \ldots, x_k)\,dx_k \ldots dx_1 \tag{13}$$

(here $\underline{x}$ is short for $(x_1, \ldots, x_k)$).

The following is essentially from [10], (Proposition 1), except that loc. cit. deals with the simplicial, rather than with the cubical, combinatorial form.

Proposition 4 *Let ω be a (cubical-combinatorial) k-form on a manifold M. Then for any $k+1$-tuple $(x_0, x_1, \ldots, x_k)$ of mutual neighbours in M,*

$$[x_0, x_1, \ldots, x_k]^*(\omega) = \omega(x_0, x_1, \ldots, x_k) \cdot \mathrm{Vol}\,.$$

From this we deduce

Proposition 5 *Let ω be a (cubical-combinatorial) k-form on a manifold M. Then for any $k+1$-tuple $(x_0, x_1, \ldots, x_k)$ of mutual neighbours in M, we have*

$$\int_{[x_0,x_1,\ldots,x_k]} \omega = \omega(x_0, x_1, \ldots, x_k).$$

Proof It is immediate that for any constant λ (like $\omega(x_0, x_1, \ldots, x_k)$), we have the second equality sign in $\int_{\mathrm{id}} \lambda \cdot \mathrm{Vol} = \int_0^1 \ldots \int_0^1 \lambda \, dt_k \ldots dt_1 = \lambda$. So the claim follows. □

Consider a k-form $\omega = g \cdot \mathrm{Vol}$ on R^k, where $g : R^k \to R$ is a function. By definition, we have $\int_{\mathrm{id}} \omega = \int_0^1 \ldots \int_0^1 g(t_1, \ldots, t_k) \, dt_k \ldots dt_1$. But we have more generally:

Proposition 6 *Consider the map $\alpha : R^k \to R^k$ given, as in (13), by $\alpha = [a_1, b_1] \times \ldots \times [a_k, b_k]$, and let $\omega = g \cdot \mathrm{Vol}$. Then*

$$\int_\alpha \omega = \int_{a_1}^{b_1} \ldots \int_{a_k}^{b_k} g(t_1, \ldots, t_k) \, dt_k \ldots dt_1. \tag{14}$$

Proof The determinant of the linear part of the affine map α is here clearly $\prod(b_i - a_i)$. From (11), we therefore have that $\int_{\mathrm{id}} \alpha^*(\omega)$ is

$$\int_0^1 \ldots \int_0^1 g([a_1, b_1](t_1), \ldots, [a_k, b_k](t_k)) \cdot \prod(b_i - a_i) \, dt_k \ldots dt_1;$$

now using (13), we see that this equals

$$\int_{a_1}^{b_1} \ldots \int_{a_k}^{b_k} g(t_1, \ldots, t_k) \, dt_k \ldots dt_1. \qquad \square$$

We can now prove:

Proposition 7 *Let θ be a k-form on a manifold M. Then the functional $\gamma \mapsto \int_\gamma \theta$ satisfies the subdivision property.*

Proof Let $\gamma : R^k \to M$ be a map (a singular k-cube). With the notation of Section 2, we need to prove that for $i = 1, \ldots, k$ and $a, b, c \in R$, we have

$$\int_{\gamma|_i[a,c]} \theta = \int_{\gamma|_i[a,b]} \theta + \int_{\gamma|_i[b,c]} \theta.$$

Let $\gamma^*(\theta) = g \cdot \mathrm{Vol}$, ($g$ a function $R^k \to R$). Then

$$\int_{\gamma|_i[a,b]} \theta = \int_{[a,b]_i} \gamma^*(\theta) = \int_0^1 \ldots \int_a^b \ldots \int_0^1 g(t_1, \ldots, t_k) \, dt_k \ldots dt_1,$$

by Proposition 6, and similarly for $\int_{\gamma|_i[a,c]}$ and $\int_{\gamma|_i[b,c]}$. The result now follows from the subdivision rule $\int_a^c = \int_a^b + \int_b^c$ for one-variable integrals (which immediately gives similar rules for subdivision of k-variable integrals). □

Some of the results proved lead to

Theorem 1

1) *For θ a k-form on a manifold M, the functional $\gamma \mapsto \int_\gamma \theta$ is an observable on M.*
2) *Let γ be an infinitesimal k-dimensional parallelepipedum; then*

$$\int_\gamma \theta = \theta(\gamma).$$

Proof The functional described has the subdivision property, by Proposition 7, and the fact that it is alternating is an immediate consequence of (11) applied to the affine map α which interchanges ith and jth coordinate. Finally, the last assertion is contained in Proposition 5. □

6 Uniqueness of Observables

We shall in this section prove that a k-dimensional observable is completely determined by its value on infinitesimal parallelepipeda. Since the k-dimensional observables on a manifold M clearly form a linear subspace of the space of all functions $M^{R^k} \to R$, it suffices to prove

Theorem 2 *Let Ψ be a k-dimensional observable which takes value* 0 *on all infinitesimal parallelepipeda. Then Ψ is constant* 0*.*

Proof Let $\gamma : R^k \to M$ be a singular cube. Consider the observable $\Phi := \gamma^*(\Psi)$ on R^k. Because any map, in particular γ, preserves infinitesimal parallelepipeda, it follows that Φ vanishes on all infinitesimal k-dimensional parallelepipeda in R^k. We shall prove that Φ vanishes on all *rectangles* in R^k; by a *rectangle* in R^k, we understand (cf. Section 1) a singular cube $\alpha : R^k \to R^k$ which is not only affine, but whose linear part is given by a *diagonal* matrix, $\alpha = [a \mid A]$ with A a diagonal matrix; the diagonal entries of this matrix are called the *sides* of the rectangle. If we can prove this, then Φ will certainly vanish on the identity map $[0 \mid I,]$ where I is the identity matrix, and $\Phi(\mathrm{id}) = \gamma^*(\Psi)(\mathrm{id}) = \Psi(\gamma)$ which therefore is 0.

So the Theorem will be proved by proving the following two Lemmas.

Lemma 1 *Assume that a k-dimensional observable Φ on R^k vanishes on all infinitesimal k-dimensional parallelepipeda in R^k. Then it vanishes on all rectangles with infinitesimal sides.*

(Note that an infinitesimal parallelepipedum need not be a rectangle; in fact, those infinitesimal k-dimensional parallelepipeda in R^k that are also rectangles have the property that *any* observable vanishes on them. See Section 7.)

Lemma 2 *Assume that a k-dimensional observable Φ on R^k vanishes on all rectangles with infinitesimal sides. Then it vanishes on all rectangles.*

Proof of Lemma 1 We prove in fact more, namely that Φ vanishes on all singular cubes of the form $[x_0, x_1, \dots, x_k]$ where $x_i \sim x_0$ for all i. (Note that such cube is not necessarily an *infinitesimal* parallelepipedum, since we have not required $x_i \sim x_j$.) There is no harm in assuming that $x_0 = 0$. The k-tuple $(x_1, \dots, x_k)$ is therefore an element of $D(k)^k$. Let $\phi : D(k)^k \to R$ be the function $(x_1, \dots, x_k) \mapsto \Phi([0, x_1, \dots, x_k])$. It is an easy consequence of the subdivision property for Φ that ϕ vanishes if one of the x_is is 0. It therefore follows from the KL axioms that ϕ extends to a k-linear map $(R^k)^k \to R$, and this map is alternating because of the alternating property assumed for Φ. By the assumption that Φ vanishes on infinitesimal parallelepipeda, it follows that ϕ vanishes on $\widetilde{D}(k,k) \subseteq D(k)^k$. But a k-linear alternating map $(R^k)^k \to R$ is completely determined by its restriction to $\widetilde{D}(k,k)$. So ϕ is the zero map. □

Proof of Lemma 2 This is by a downward induction, starting from k: by assumption, Φ vanishes on rectangles with all k sides infinitesimal. Assume we have already proved that Φ vanishes on all rectangles with the i first sides infinitesimal; we prove that it then also vanishes on rectangles with the $i-1$ first sides infinitesimal. Consider such a rectangle $[x_0, x_0 + t_1e_1, \dots, x_0 + t_{i-1}e_{i-1}, x_0 + t_ie_i, \dots, x_0 + t_ke_k]$, where the $t_1, \dots, t_{i-1}$ are in D. Consider this as a function of t_i alone, in other words, consider the function $g : R \to R$ given, for fixed x_0 and t_j $(j \neq i)$, by

$$g(t) := \Phi([x_0, x_0 + t_1e_1, \dots, x_0 + t_{i-1}e_{i-1}, x_0 + te_i, \dots, x_0 + t_ke_k]).$$

If $t = 0$, the input to Φ is a rectangle with i infinitesimal sides, and so $g(0) = 0$.

Let us spell out $g(t)$ in the matrix notation for affine maps;

$$g(t) = \Phi\left(\left[\begin{array}{c|cccc} x_{01} & t_1 & & & \\ \vdots & & \ddots & & \\ x_{0i} & & & t & & \\ \vdots & & & & \ddots & \\ x_{0k} & & & & & t_k \end{array}\right]\right).$$

By the subdivision of rectangles exhibited in (8), it is now clear that $g(t+d) - g(t)$ equals

$$\Phi\left(\left[\begin{array}{c|cccc} x_{01} & t_1 & & & \\ \vdots & & \ddots & & \\ x_{0i} + t & & & d & & \\ \vdots & & & & \ddots & \\ x_{0k} & & & & & t_k \end{array}\right]\right)$$

The input of Φ here is a rectangle with its first i sides infinitesimal, so by the induction assumption, the value of Φ on it is 0. Thus $g(t+d) = g(t)$ for all $d \in D$, i.e. $g' \equiv 0$.

Since also $g(0) = 0$, we conclude that $g \equiv 0$. So Φ vanishes on all rectangles with basic vertex in x_0, but x_0 was arbitrary; so Φ vanishes on all rectangles. This proves the Lemma. (This is essentially the argument given already in [15], after Lemma 4.4 in there, and reproduced in [7] and [16].) □

With these Lemmas, the theorem follows. □

Proposition 8 *Every k-dimensional observable Φ is of the form $\int \omega$ for a unique k-form ω. In particular, the process $\omega \mapsto \int \omega$ is a bijection between k-forms and k-dimensional observables.*

Proof If such an ω exists, then by the second assertion in Theorem 1, we have for any infinitesimal parallelepipedum $[x_0, \ldots, x_k]$ that

$$\omega(x_0, \ldots, x_k) = \Phi([x_0, \ldots, x_k]);$$

this proves that there is at most one k-form ω which gives rise to Φ by integration. Conversely, let us define ω by this formula. To see that the ω thus defined is indeed a differential form, we have to see that it vanishes on degenerate infinitesimal parallelepipeda. So suppose the parallelepipedum is degenerate by virtue of $x_i = x_0$. Then it is easy to see that this parallelepipedum γ subdivides in the ith direction into two copies of itself, and so it follows from the subdivision property for Φ that $\Phi(\gamma) = \Phi(\gamma) + \Phi(\gamma)$, whence $\Phi(\gamma) = 0$, so ω does get value 0 on this parallelepipedum.

The fact that for this form ω, $\int \omega = \Phi$ follows from Proposition 5 and from the uniqueness in Theorem 2. □

Consider now a k-form ω on a manifold M, and its coboundary (= exterior derivative) $d\omega$. Since $d\omega$ is a $k+1$-form, it defines a $k+1$-dimensional observable on M, $\gamma \mapsto \int_\gamma d\omega$, γ a singular $k+1$-cube. There is another functional on the set of singular $k+1$-cubes, namely $\gamma \mapsto \int_{\partial\gamma} \omega$, where $\partial\gamma$ is a k-chain in the chain complex described in Section 3. This is, in the notation from there, the coboundary of the cochain $\int \omega$, and since $\int \omega$ is an observable, then so is $d \int \omega$, by Proposition 2.

Theorem 3 (Stokes' Theorem) *Let γ be a singular $k+1$ cube in a manifold, and let ω be a k-form on M. Then*

$$\int_{\partial\gamma} \omega = \int_\gamma d\omega.$$

Proof As functions of $\gamma \in S_{k+1}(M)$ (the set of singular $k+1$-cubes), both sides are observables (alternating and with subdivision property). On infinitesimal parallelepipeda, the two sides agree, because d for the cochain complex of differential forms was defined in terms of the ∂ in the chain complex of infinitesimal parallelepipeda. The result therefore follows from uniqueness of observables (Theorem 2). □

Some aspects of the theory presented here may be summarized: the cubical complex of infinitesimal parallelepipeda in M is a subcomplex of the cubical complex of singular cubes. In the associated cochain complexes, the differential forms constitute a subcomplex of the first, and the observables constitute a subcomplex of the second cochain complex. The inclusion map of the complex of infinitesimal parallelepipeda into the complex of singular cubes induces a bijection between k-forms and k-observables. And these bijections, as k ranges, are compatible with the coboundary operators, so that the two cochain complexes are isomorphic.

This line of reasoning and the proofs presented in this Section run in parallel with the one in Félix and Lavendhomme's [4] (see also [16] 4.5); however, they deal with the a somewhat different synthetic notion of differential form, based on "marked microcubes", as described in the next Section.

7 Comparison with other Differential Form Notions

We shall compare the treatment of differential forms both with the classical treatment, with the standard synthetic treatments, and with the simplicial theory of combinatorial forms.

In all cases, a differential form on a space M (a manifold, say) is an R-valued function which takes a suitable kind of infinitesimal entities on M as inputs. A classical differential k-form on a manifold M thus takes k-tuples of tangent vectors (with common base point) as inputs. This can immediately be paraphrased synthetically, cf. [7] Def. I.14.1, (and in this context, the theory applies not just to finite dimensional manifolds, but to arbitrary microlinear spaces, see [7] I.14 or [16] Chapter 4, say). For some other synthetic treatments, as in [7, 15] Def. I.14.2, [16] Def. 4.1.1, the inputs of a differential form are rather the "k-tangents" or the "k-microcubes" in M, meaning maps $\tau : D^k \to M$. The problem with these notions is that there is no *geometric* way to associate cubical boundary faces, neither to a k-tuple of tangents, nor to a microcube. More concretely (for $k = 1$, say), for a tangent $\tau : D \to M$, one naturally puts $\partial^0(\tau) := \tau(0) \in M$, but there is no particular $d \in D$ to give $\partial^1(\tau)$ by putting $\partial^1(\tau) := \tau(d)$.

To be able to assign a boundary to a k-microcube $D^k \to M$, the treatments of [7, 15] I.14, [4, 18] IV.1, [16] 4.2.1 all take resort to what [16] calls *marked* microcubes (and we shall follow this terminology here); these are pairs $(\tau, \underline{d})$, where $\tau : D^n \to M$ is a microcube and $\underline{d} = (d_1, \ldots, d_k) \in D^k$. Then $2k$ boundary faces $\partial_i^\alpha(\tau, \underline{d})$ can naturally be defined as marked $k-1$-microcubes. This leads to a cubical complex of marked microcubes. The cochain complex of differential forms will then, in this set up, be a subcomplex of the cochain complex associated to the cubical complex of marked microcubes. The conditions which qualify a cochain as a differential form is now that it is alternating, and satisfies homogeneity conditions, both with respect to the input entity τ, and with respect to the input entity $\underline{d}$.

The many requirements to be put on a cochain in order to qualify as a differential form may be said to have their source in the fact that a marked microcube contains an amount of redundant information in so far as differential forms go. In fact, there is a bijection between the set of differential k-forms $\overline{\omega}$ in the sense of marked microcubes, and the set of differential forms ω, in the sense of just microcubes; ω is characterized by

$$d_1 \cdot \ldots \cdot d_k \cdot \omega(\tau) = \overline{\omega}(\tau, (d_1, \ldots, d_k)),$$

for all microcubes τ and all $\underline{d} = (d_1, \ldots, d_k) \in D^k$. From this, it is easy also to prove the following for any differential form $\overline{\omega}$ (in the sense of cochain on the complex of marked microcubes):

Proposition 9 *Let $(\tau, \underline{d})$ and $(\tau', \underline{d}')$ be two marked microcubes in a manifold M, such that $\tau(0) = \tau'(0)$ and $\tau(d_i) = \tau'(d_i')$ for $i = 1, \ldots, k$, and let $\overline{\omega}$ be a differential k-form on M. Then $\overline{\omega}(\tau, \underline{d}) = \overline{\omega}(\tau', \underline{d}')$.*

It follows that differential k-forms could be defined by taking as their inputs $k+1$-tuples of points in M which come about from marked microcubes $(\tau, \underline{d})$ as $(\tau(0), \tau(d_1e_1), \ldots, \tau(d_ke_k))$ where e_i is the ith canonical basis vector for R^k; however, this $k+1$ tuple is in general not an infinitesimal k-simplex in M, since there is no reason why $\tau(d_ie_i)$ should be $\sim \tau(d_je_j)$. Therefore the affine combinations in M needed to form an infinitesimal singular parallelepipedum from them are not well defined, and the cubical faces ∂_i^1 depend on the formation of these singular cubes. (Of course, the requisite affine combinations can be formed in R^k, and then transported to M using (an extension of) τ, but then the result depends on τ.)

It is actually the case that an infinitesimal parallelepipedum in R^k ($k \geq 2$) which is of the form $[0, d_1e_1, \ldots, d_ke_k]$ is trivial in so far as differential k-forms go; this follows from the fact that the determinant of any diagonal matrix in $\widetilde{D}(k, k)$ is 0.

8 Outlook into Higher Connection Theory

The notion of "cubical complex" has obvious truncations at each dimension p (disregarding all the structure in dimensions $> p$). We call such a truncated object a *p-cubical complex*. A 1-cubical complex is the same thing as a reflexive graph. Any groupoid has an underlying 1-cubical complex.

If $\mathbf{C} = (C_1 \rightrightarrows M)$ is a groupoid with a manifold M as its set of objects, a *connection* in $\mathbf{C}$ may be construed as a morphism of reflexive graphs (fixing M) from $M_{[1]} \rightrightarrows M$ to $C_1 \rightrightarrows M$, cf. [8] and [11]. (Note that the combinatorial encoding of differential geometric notions in terms of the neighbour relation makes it unnecessary to consider, alongside with the groupoid itself, linearized infinitesimal versions of it, like some kind of "(Lie-) algebroid". This also applies to higher groupoids/connections.)

If $C_1 \rightrightarrows M$ is the "constant" groupoid on M given by the additive group $(R, +)$ (so the arrows in it from x to y are triples (x, a, y) with $a \in R$, and composition given by addition of the as), then a connection in this groupoid may be identified with a combinatorial 1-form on M.

We may utilize our understanding of (cubical-combinatorial) differential forms to arrive at a notion of a higher connection on M, with values in a "cubical groupoid" on M. We shall sketch this in the dimension 2.

We first note that there is a forgetful functor from 2-cubical complexes to 1-cubical complexes. This functor has adjoints on both sides. We shall describe the right adjoint $\mathbf{C} \mapsto \mathbf{C}'$ in case the 1-cubical complex is the underlying one of a groupoid $\mathbf{C}$. The right adjoint then gives the double groupoid $\mathbf{C}'$ whose 0- and 1-cells agree with those of $\mathbf{C}$ and whose 2-cells are the (not necessarily commutative) squares in $\mathbf{C}$. It is a "special" or "edge symmetric" double groupoid ([3]), meaning that the set of "horizontal" and "vertical" arrows agree.

Now consider a connection ∇ in a groupoid $\mathbf{C} = (C_1 \rightrightarrows M)$. By the universal property of right adjoints, we get a morphism $\widehat{\nabla}$ of (the 2-truncation of) $M_{[\bullet]}$ into $\mathbf{C}'$. This $\widehat{\nabla}$ agrees with ∇ in dimension 1, and in dimension 2, it associates to an infinitesimal singular parallelogram $\gamma = [x_0, x_1, x_2]$ in M the (not-necessarily commutative) square in the groupoid

$$\begin{tikzcd} x_2 \arrow[r, "\nabla(x_2,u)"] & u \\ x_0 \arrow[u, "\nabla(x_0,x_2)"] \arrow[r, "\nabla(x_0,x_1)"'] & x_1 \arrow[u, "\nabla(x_1,u)"'] \end{tikzcd}$$

where u is $x_1 - x_0 + x_2 = \partial^1(\partial^1_1(\gamma))$ (the fourth corner in the parallelogram spanned by x_0, x_1, x_2; this makes sense by the assumption that (x_0, x_1, x_2) form an infinitesimal 2-simplex).

This 2-cell in $\mathbf{C}'$ we call (cf. [14]) (the value at (x_0, x_1, x_2) of) the *formal curvature* of ∇. It is an example of a 2-connection in the edge symmetric double groupoid $\mathbf{C}'$, in the sense of loc.cit. We shall not here go further into the theory from loc.cit. of higher cubical groupoids, the higher connections in them, curvature = coboundary, holonomy = integrals, Stokes' Theorem etc., but only argue why formal curvature of a 1-connection ∇ agrees with coboundary of the corresponding cubical 1-form, in case the groupoid $\mathbf{C}$ is the constant groupoid given by $(R, +)$ (considered above). The relationship between ∇ and ω is

$$\nabla(x_0, x_1) = (x_0, \omega(x_0, x_1), x_1).$$

Now a square in a groupoid gives rise to an endo-arrow at the (first, say) vertex of the groupoid, by taking cyclic composite. In our case, this cyclic composite gives

$$\omega(x_0, x_1) + \omega(x_1, u) - \omega(x_2, u) - \omega(x_0, x_2),$$

which is just $d\omega$ applied to the infinitesimal parallelogram γ.

For higher connections, say 2-connections in an edge-symmetric double groupoid $\mathbf{C}$, one needs similarly to describe cyclic composites of the faces of a cube (see [2]) in the 3-cubical groupoid $\mathbf{C}'$; this is more delicate, and requires a "folding" composite of 2-cells in the 3-cubical groupoid, which the simplicial account of combinatorial forms does not seem to provide.

The bijection between forms and observables, for $k = 1$, may be generalized into a bijection between connections and path-connections, (holonomies), see [11], and this can be generalized to higher connections and their holonomies, [14].

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