CALCULUS OF SMOOTH FUNCTIONS BETWEEN CONVENIENT VECTOR SPACES

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Dedicated to the memory of Svend Bundgaard

We give an exposition of some basic results concerning remainders in Taylor expansions for smooth functions between convenient vector spaces, in the sense of Frölicher and Kriegl, cf. [2], [11], [3], [13].

We needed such results in [9], but could not find them in the works quoted.[1]

The method we employ is very puristic: we never have to consider limits, or any other analytic tools, except for finite dimensional vector spaces $\mathbf{R}^n$. In this sense, we carry Frölicher's program of considering mutually balancing sets of curves $\mathbf{R} \to X$ and functions $X \to \mathbf{R}$ to the extreme. (Also, the puristic aspect makes it easy to transfer the theory to "synthetic" contexts, like [7].)

Besides the introduction, where we recall some of the existing theory, the paper contains two sections: 1) on the general theory of Taylor remainders for smooth maps $X \to Y$, where X and Y are convenient vector spaces; and 2) a more refined theory for the case where $X = \mathbf{R}^n$.

First, we recall the notion of convenient vector space in the formulation of [3]: it is a vector space X over $\mathbf{R}$, equipped with a linear subspace X'

*This is a retyping of a preprint [8] with the same title, Aarhus Preprint Series 1984/85 No. 18. The bibliography has been updated, since [9] and [10] in the meantime have been published. Also, [4] has been published (1988). The numbering of the equations have changed, but the numbering of Propositions, Theorems, etc. is unchanged compared to the Preprint Version.

[1] [4] does have some of these results; [8] is quoted there (Section 4.4) in connection with Taylor expansion.

of the full algebraic dual X^*, such that X' separates points, and with the following two completeness properties:

- The bornology induced on X by X' is a complete bornology
- with respect to this bornology, every bounded linear $X \to \mathbf{R}$ belongs to X'

We refer the reader to [6], say, for the bornological notions, but actually we shall not directly use these completeness properties, but rather some consequences of them, contained in particular in the following results below: Proposition 0.3, Theorem 0.7, and Theorem 2.1 .

In the following X, Y, Z etc. denote always convenient vector spaces, $X = (X, X')$, etc. The vector space $\mathbf{R}^n$ carries a unique convenient structure, namely the full linear dual.

The following notions are basic in the approach of Frölicher [2]:

Definition 0.1 *A map $c : \mathbf{R}^n \to X$ is called* smooth *if for all $\phi \in X'$, $\phi \circ c : \mathbf{R}^n \to \mathbf{R}$ is smooth ($= C^\infty$).*

If $X = \mathbf{R}^m$, this is clearly consistent with the standard use of the word 'smooth'.

Definition 0.2 *A map $f : X \to Y$ is called* smooth *if for any smooth $c : \mathbf{R}^n \to X$, $f \circ c$ is smooth.*

This is clearly consistent with Definition 0.1. Also, smooth maps compose, and so convenient vector spaces and smooth maps form a category $\underline{F}$

Proposition 0.3 *The smooth linear maps $X \to \mathbf{R}$ are exactly the elements of X'.*

Proof. See [3] Proposition 3.1.

Proposition 0.4 *If X and Y are convenient, then $X \oplus Y = X \times Y$ carries a convenient structure $(X \oplus Y)'$, defined as $X' \oplus Y' \subseteq X^* \oplus Y^* = (X \oplus Y)^*$. With this structure, $X \oplus Y$ is the categorical product in the category $\underline{F}$.*

Proof. See remark after Theorem 6.2 in [3].

Let $C^\infty(X,Y)$ denote the set of smooth maps $X \to Y$. It carries a natural vector space structure induced from that of Y.

Theorem 0.5 *There is a convenient structure on $C^\infty(X,Y)$ making it into an exponential object Y^X in $\underline{F}$, (which thus is cartesian closed).*

This is a main result in [12], [2], and a main motivation for their theory. – With this convenient structure on $C^\infty(X,Y)$, a map $Z \to C^\infty(X,Y)$ is smooth iff its transpose $Z \times X \to Y$ is smooth.

Let $L(X,Y) \subseteq C^\infty(X,Y)$ denote the linear subspace of smooth linear maps. It carries a convenient structure in such a way that $Z \to L(X,Y)$ is smooth iff $Z \to L(X,Y) \subseteq C^\infty(X,Y)$ is smooth. Similarly

$$L^p(X,Y) \subseteq C^\infty(\underbrace{X \times \ldots \times X}_{p} \to Y)p$$

for the linear subspace of smooth p-linear maps $X \times \ldots \times X \to Y$. In particular $L(X,\mathbf{R}) = X'$, by Proposition 0.3. So X' acquires itself a convenient structure.

Remark 0.6 In general, linear and multilinear maps need not be smooth. However, linear maps $f : \mathbf{R}^n \to X$ are automatically smooth, since for $\phi \in X'$, $\phi \circ f : \mathbf{R}^n \to \mathbf{R}$ is linear, hence smooth. Similarly for multilinear maps. Also, for $t \in \mathbf{R}$, the homothety 'multiplication by t' :$X \to X$ is smooth, and so is any 'translation by a constant', $x \mapsto x + a$. And any map $X_1 \oplus X_2 \to Y_1 \oplus Y_2$ defined by a 2×2 matrix of smooth linear maps $f_{ij} : X_i \to Y_j$ is smooth linear. Finally, for $x_0, x_1 \in X$, the map (which we shall use later),

$$[x_0, x_1] : \mathbf{R} \to X, \tag{1}$$

defined by $t \mapsto x_0 + t \cdot x_1$, is smooth.

– All these remarks are verified the same standard way: test with relevant smooth linear functionals ϕ into $\mathbf{R}$ (and with smooth 'plots' $c : \mathbf{R}^n \to X$, is case X is not finite dimensional), to reduce the question to the finite dimensional case, where the result is well known.

Note that if $\xi : X \to Y$ is linear, then

$$\xi \circ [x_0, x_1] = [\xi(x_0), \xi(x_1)]. \tag{2}$$

Let $f : X \to Y$ be smooth. For any $x_0, x_1 \in X$ and $\phi \in Y'$, consider the smooth function

$$\mathbf{R} \xrightarrow{[x_0, x_1]} X \xrightarrow{\quad f \quad} Y \xrightarrow{\quad \phi \quad} \mathbf{R} \tag{3}$$

(cf. (1) for notation). A main result in [3] is

Theorem 0.7 *For any smooth $f : X \to Y$, there exists a (unique) smooth function*

$$X \xrightarrow{df} L(X, Y)$$

such that for any x_0, x_1 and $\phi \in Y'$,

$$\phi(df(x_0; x_1)) = (\phi \circ \ f \circ [x_0, x_1])'(0).$$

(We write $df(x_0; x_1)$ instead of $df(x_0)(x_1)$, and $(\ )'$ denotes ordinary derivative of smooth functions $\mathbf{R} \to \mathbf{R}$, in particular (3).

For purely algebraic reasons, we get the following 'linear' chain rules for df:

Proposition 0.8 *Let $f : X \to Y$ be smooth. If $\psi : Y \to Z$ is smooth linear*

$$d(\psi \circ f)(x_0; x_1) = \psi(df(x_0; x_1)), \tag{4}$$

and if $\xi : Z \to X$ is smooth linear

$$d(f \circ \xi)(z_0; z_1) = df(\xi(z_0); \xi(z_1)). \tag{5}$$

Proof. Let $\phi \in Z'$ be arbitrary. Then

$$\phi(d(\psi \circ f)(x_0; x_1)) = (\phi \circ \psi \circ f \circ [x_0, x_1])'(0)$$
$$= (\phi \circ \psi)(df(x_0; x_1))$$

(since $\phi \circ \psi \in Y'$)

$$= \phi(\psi(df(x_0; x_1)).$$

Since this holds for all $\phi \in Z'$, and Z' separates points, we conclude (4). Similarly, for (5), let $\phi \in Y'$ be arbitrary. Then

$$\phi(d(f \circ \xi)(z_0; z_1)) = d(\phi \circ f \circ \xi)(z_0, z_1)$$

(by (4))

$$= (\phi \circ f \circ \xi \circ [z_0, z_1])'(0)$$
$$= (\phi \circ f \circ [\xi(z_0), \xi(z_1)])'(0)$$

(by (2))

$$= \phi(df(\xi(z_0); \xi(z_1));$$

and since Y' separates points, we conclude (5).

For $p \in \mathbf{N}$, and $f : X \to Y$ smooth, one defines inductively a smooth

$$d^p f : X \to L^p(X, Y)$$

as the composite

$$X \xrightarrow{d(d^{p-1}f)} L(X, L^{p-1}(X, Y)) \cong L^p(X, Y).$$

So for $x_0, \ldots, x_p \in X$, we have $d^p f(x_0; x_1, \ldots, x_p)$ depending smoothly on the x_i's simultaneously, and linearly on those after the semicolon.

Proposition 0.8 (p) *Let f, ψ, ξ be as in Proposition 0.8. Then*

$$d^p(\psi \circ f)(x_0; x_1, \ldots, x_p) = \psi(d^p f(x_0; x_1, \ldots, x_p)) \tag{6}$$

and

$$d^p(f \circ \xi)(z_0; z_1, \ldots, z_p) = d^p f(\xi(z_0); \xi(z_1), \ldots, \xi(z_p)). \tag{7}$$

Also, if $\xi : X \to X$ is translation by a constant,

$$d^p(f \circ \xi)(z_0; z_1, \ldots, z_p) = d^p f(\xi(z_0); z_1, \ldots, z_p). \tag{8}$$

Proof. Easy induction in p. We shall do (7) only. Assume (7) for n (for $p = 1$, the case is settled by (5)). Let ev_x denote the smooth linear map 'evaluating at $x_2, \ldots, x_{p+1}$'

$$ev_x : L^p(X, Y) \to Y.$$

For any $\phi \in Y'$, consider $\phi \circ ev_x \in (L^p(X, Y))'$. Then the defining property of d^{p+1} in terms of $d(d^p)$ yields

$$\phi(d^{p+1} f(x_0; x_1, \ldots, x_{p+1})) = (\phi \circ ev_x \circ d^p f \circ [x_0, x_1])'(0). \tag{9}$$

Thus, for $z = (z_2, \ldots, z_{p+1})$ and $\xi(z) = (\xi(z_2), \ldots, \xi(z_{p+1}))$, we have by (0.11) (with z_i for x_i, $f \circ \xi$ for f)

$$\phi(d^{p+1}(f \circ \xi)(z_0; z_1, \ldots, z_{p+1}))$$

$$= (\phi \circ ev_z \circ d^p(f \circ \xi) \circ [z_0, z_1])'(0)$$
$$= (\phi \circ ev_{\xi(z)} \circ d^p(f) \circ [\xi(z_0), \xi(z_1)])'(0)$$

(by induction, and by (2)),

$$= \phi(d^{p+1} f(\xi(z_0); \xi(z_1), \ldots, \xi(z_{p+1}))$$

by (9) again. Since this holds for all $\phi \in Y'$, and Y' separates points, we get (7) for $p + 1$.

Corollary 0.9 *Let $f : X \to Y$ be smooth. For fixed $x_0 \in X$, $d^p f(x_0; x_1, \ldots, x_p)$ is symmetric in the remaining p arguments.*

Proof. Using (8), the problem reduces to the case $x_0 = 0$. Using (6) with ψ ranging over Y' reduces the problem to the case where $Y = \mathbf{R}$. Let σ be a permutation of $\{1, \ldots, p\}$, and let $\Sigma : \mathbf{R}^p \to \mathbf{R}^p$ be the corresponding linear map, $e_i \mapsto e_{\sigma(i)}$, where e_i is the i'th canonical basis vector. Also, consider the (smooth) linear map $\xi : \mathbf{R}^p \to X$ given by $e_i \mapsto x_i, (i = 1, \ldots, p)$. We now use (7) in connection with the smooth linear maps ξ, Σ, and $\xi \circ \Sigma$:

$$d^p f(0; x_1, \ldots, x_p) = d^p(f \circ \xi)(0; e_1, \ldots, e_p)$$

(by (7) for ξ)

$$= d^p(f \circ \xi)(0; e_{\sigma(1)}, \ldots, e_{\sigma(p)})$$

(by the known symmetry in the finite dimensional case ([1], (8.2.4)),

$$= d^p(f \circ \xi \circ \Sigma)(0; e_1, \ldots, e_p)$$

(by (7) for Σ)

$$= d^p f(0; x_{\sigma(1)}, \ldots, x_{\sigma(p)})$$

(by (7) for $\xi \circ \Sigma$). This proves the Corollary.

Let us finally note

Proposition 0.10 *The process $f \mapsto df$ defines a (smooth) linear map $d : C^\infty(X, Y) \to C^\infty(X, L(X, Y))$.*

Proof. Using (6) with ψ ranging over Y' easily reduces the linearity question to the finite dimensional case, where it is known. The smoothness we shall not need; for a proof, see [12] Satz p. 299.

1 General theory of Taylor remainders

For a smooth $f : \mathbf{R} \to \mathbf{R}$, the following two conditions are known to be equivalent, and if they are satisfied, f is said to have order $\geq k$:

- There exists a smooth $\mathbf{R} \to \mathbf{R}$ such that $f(t) = t^k \cdot g(t)\ \forall t$;
- $f(0) = f'(0) = ... = f^{(k-1)}(0) = 0$

(for the equivalence, cf. e.g. [1], Exercise 6 in VIII.14).

In the following, $X, Y, Z, \ldots$ denote convenient vector spaces, as before.

Definition 1.1 *A smooth map $f : X \to Y$ is of* order $\geq k$ *if for every $\phi \in Y'$ and every $x \in X$, the function $\mathbf{R} \to \mathbf{R}$ given by $t \mapsto \phi(f(t \cdot x))$ is of order $\geq k$.*

For an equivalent definition, see Theorem 2.13 below.

Clearly these functions form a linear subspace of $C^\infty(X, Y)$, which we denote $Ord_k(X, Y)$.

Theorem 1.2 *Let $f : X \to Y$ be smooth. For any $x_0 \in X$, the function $g : X \to Y$ defined by*

$$g(x) := f(x_0 + x) - \sum_{p=0}^{k-1} \frac{1}{p!} d^p f(x_0; x, \ldots x) \tag{10}$$

is of order $\geq k$.

Proof. For any $\phi \in Y'$, we have, by Proposition 0.8 that

$$\phi(g(t \cdot x)) = (\phi \circ f \circ [x_0, x])(t) - \sum_{p<k} \frac{1}{p!} d^p(\phi \circ f \circ [x_0, x])(0; t, \ldots, t).$$

The sum here is the k first terms of the Taylor series at 0 of the function $\phi \circ f \circ [x_0, x] : \mathbf{R} \to \mathbf{R}$, and therefore the difference is of order $\geq k$ as a function of t.

Let us call a smooth map $f : X \to Y$ *p-homogeneous* if $f(t \cdot x) = t^p \cdot f(x)$, $\forall t \in \mathbf{R}$, $x \in X$, and let $Poly_{<k}(X, Y)$ denote the linear subspace of $C^\infty(X, Y)$ spanned by p-homogeneous maps for all $p < k$, ('polynomial maps of degree $< k$'). We easily see that

$$Poly_{<k}(X, Y) \cap Ord_k(X, Y) = \{0\}; \tag{11}$$

for if $f : X \to Y$ belongs to both, then for any $\phi \in Y'$

$$(\phi \circ f)(t \cdot x) = \sum_{p<k} t^p \cdot g_p(x) = t^k \cdot h(t, x)$$

for suitable h, smooth in t. Keeping x fixed, this, by a standard finite dimensional result, means that $(\phi \circ f)(t \cdot x) = 0$ $\forall t$, hence $(\phi(f(x)) = 0$. Since x and ϕ are arbitrary, and Y' separates points, $f(x) \equiv 0$.

Now consider (10) for $x_0 = 0$. The sum $\sum_{p<k}$ there is clearly polynomial of degree $< k$, so from (11) and Theorem 1.2, we obtain the two first assertions in

Corollary 1.3 *1) Every smooth function $f : X \to Y$ can uniquely be written $f = h + g$, with g of order $\geq k$, and h polynomial of degree $< k$; equivalently*

$$C^\infty(X, Y) = Poly_{<k}(X, Y) \oplus Ord_k(X, Y).$$

2) Any polynomial map $h : X \to Y$ of degree $< k$ can be written in the form

$$h(x) = \sum_{p<k} h_p(x, ...x) \tag{12}$$

with $h_p : X\times, \ldots, \times X \to Y$ p-linear and smooth. 3) The decomposition (12) is unique if we require the h_p's to be symmetric.

Proof of 3). Since a k-homogeneous map is of order $\geq k$, it follows by an easy downward induction on p, using (11), that the p-homogeneous part of a smooth map is uniquely determined. So it suffices to see that if h is smooth p-linear, there exist a unique symmetric smooth p-linear map $\tilde{h}$ with

$$h(x, \ldots, x) = \tilde{h}(x, \ldots, x)\ \forall x.$$

But the purely algebraic symmetrization process for p-linear maps preserves the property of being smooth (being essentially a finite linear combination). Also, the purely algebraic process which to a symmetric p-linear map h associates its diagonalization $h(x, \ldots, x)$ is injective, since h can be reconstructed from the latter by polarization, cf. [5] 9.12.

Corollary 1.4 *A smooth k-homogeneous map $f : X \to Y$ is of form $f(x) = h(x, ...x)$ for a unique smooth k-linear symmetric $h : X^k \to Y$. In particular, 1-homogeneous smooth maps are linear.*

2 Taylor remainders for smooth plots

For any convenient vector space X, we have a linear bijection $ev : L(\mathbf{R}, X) \to X$ given by "evaluation at $1 \in \mathbf{R}$". (It is also, for quite trivial reasons, an isomorphism for the convenient structures, but we do not use that fact.) For any smooth map $f : \mathbf{R} \to X$ ('a smooth curve'), we define its derivative $f' : \mathbf{R} \to X$ to be the composite

$$\mathbf{R} \xrightarrow{df} L(\mathbf{R}, X) \xrightarrow{ev} X.$$

From linearity of the operator d (Proposition 0.10) and of ev, one immediately sees that $f \mapsto f'$ is a linear map

$$C^\infty(\mathbf{R}, X) \to C^\infty(\mathbf{R}, X)$$

(it is in fact smooth linear). Also, if $\phi : X \to Y$ is smooth linear, (6) implies

$$(\phi \circ f)' = \phi \circ f'. \tag{13}$$

So if $f' \equiv 0$, we have $(\phi \circ f)' \equiv 0$ for all $\phi \in Y'$, and so $\phi \circ f : \mathbf{R} \to \mathbf{R}$ is constant. Since Y' separates points, f itself must be constant. So a smooth curve $f : \mathbf{R} \to X$ can, modulo a constant, have at most one primitive, i.e. a smooth curve $g : \mathbf{R} \to X$ with $g' \equiv f$. A fundamental completeness for convenient vector spaces is now that for every smooth curve, there is in fact a primitive, cf. [11] Satz p. 119; so

Theorem 2.1 *Every smooth curve $\mathbf{R} \to X$ has a primitive, unique up to a constant $\in X$.*

Definition 2.2 *Let $\mathbf{R} \to X$ be smooth, and $a, b \in \mathbf{R}$. Then*

$$\int_a^b f(s)\, ds := g(b) - g(a) (\in X)$$

where g is any primitive of f.

From (13) and the definition immediately follows that

Proposition 2.3 *Let $\phi : X \to Y$ be smooth linear. Then*

$$\phi(\int_a^b f(s)\, ds) = \int_a^b \phi(f(s))\, ds.$$

We also have

Proposition 2.4 *Let $f : \mathbf{R} \to X$ be smooth. Then the function $g : \mathbf{R} \to X$ defined by*

$$g(t) := \int_a^t f(s)\, ds$$

is smooth and satisfies $g' \equiv f$.

Proof. To see g is smooth, we must (definition 0.1) check that $\phi \circ g$ is smooth $\forall \phi \in X'$. But by Proposition 2.3,

$$\phi(g(t)) = \int_a^t (\phi \circ f)(s)\, ds \tag{14}$$

and since $\phi \circ f$ is smooth, this integral is, by known finite dimensional calculus, smooth as a function of t. Also, by the fundamental theorem of calculus, (14) gives $(\phi \circ g)' = \phi \circ f$. On the other hand, by (13), $(\phi \circ g) = \phi \circ g'$. So $\phi \circ g' = \phi \circ f \; \forall \phi \in X'$, and since separates points, $g' = f$.

Exactly the same technique proves the following three Propositions:

Proposition 2.5 *For fixed $a, b \in \mathbf{R}$, $\int_a^b f(s)\, ds$ depends linearly on $f \in C^\infty(\mathbf{R}, X)$.*

Proposition 2.6 *Let $h : \mathbf{R}^n \times \mathbf{R} \to X$ be smooth. Then the function $g : \mathbf{R}^n \to X$ defined by*

$$g(\underline{t}) := \int_a^b h(\underline{t}, s)\, ds$$

is smooth.

(These results can be interpreted as saying: $\int_a^b$ is a smooth linear function $C^\infty(\mathbf{R}, X) \to X$.)

Proposition 2.7 *If $h : \mathbf{R} \to \mathbf{R}$ and $f : \mathbf{R} \to X$ are smooth, then*

$$(f \circ h)' = (f' \circ h) \cdot h'. \tag{15}$$

Theorem 2.8 *Assume $f : \mathbf{R} \to X$ is smooth curve with $f(0) = 0$. Then there exist a unique smooth curve $g : \mathbf{R} \to X$ such that $f(t) = t \cdot g(t) \; \forall t \in \mathbf{R}$.*

Proof. Define g by

$$g(t) = \int_0^1 f'(t \cdot s)\, ds. \tag{16}$$

Smoothness of g then follows from Proposition 2.6, with $h(t, s) = f'(t \cdot s)$. To prove (15), let $\tau : \mathbf{R} \to \mathbf{R}$ be multiplication by t (so $\tau' = t$). We then have

$$\begin{aligned} t \cdot g(t) = t \cdot \int_0^1 f'(t \cdot s)\, ds &= \int_0^1 t \cdot f'(t \cdot s)\, ds \\ = \int_0^1 (f \circ \tau)'(s)\, ds &= (f \circ \tau)(1) - (f \circ \tau)(0), \end{aligned}$$

using Proposition 2.1 and 2.7, and the Definition 2.2 of integrals. But $f(\tau(1)) = f(t)$, and $f(\tau((0)) = f(0) = 0$. – To prove uniqueness of g, it suffices to prove that if $t \cdot g(t) \equiv 0$ for a smooth curve $g : \mathbf{R} \to X$, then $g \equiv 0$. This is known for the case $X = \mathbf{R}$, and the general result now follows in the standard way, using that X' separates points.

Corollary 2.9 *Any smooth curve* $f : \mathbf{R} \to X$ *of order* $\geq k$ *can be written* $f(t) = t^k \cdot g(t)$ *for a unique smooth* $g : \mathbf{R} \to X$.

Proof. By induction in k. The theorem gives the case $k = 1$. Assume the result for $k - 1$. If f has order $\geq k$, it has order $> k - 1$, and thus can be written

$$f(t) = t^{k-1} \cdot h(t).$$

It suffices to prove that $h(0) = 0$, and then apply the Theorem to h. Now, for any $\phi \in X'$

$$t^{k-1} \cdot (\phi \circ h)(t) = (\phi \circ f)(t) = t^k \cdot g_\phi(t)$$

for some smooth $g_\phi : \mathbf{R} \to \mathbf{R}$, by the assumption on f. Now since $\phi \circ h$ and g_ϕ are smooth functions $\mathbf{R} \to \mathbf{R}$, we get by this equation that $(\phi \circ h)(0) = 0$ (standard result about smooth, in fact about continuous, mappings $\mathbf{R} \to \mathbf{R}$). Since X' separates points, $h(0) = 0$. – Uniqueness is proved like the uniqueness assertion in the theorem.

An analogous result for several variables is Theorem 2.11 below. To prove that, we first need the following reformulation of previous results.

Proposition 2.10 *Any smooth function* $f : \mathbf{R} \to Y$ *can, for given* k, *be written*

$$f(t) = \sum_{p<k} t^p \cdot y_p + t^k \cdot g(t)$$

for unique $y_p \in Y$ *and unique smooth* $g : \mathbf{R} \to Y$.

Proof. A p-linear map $\mathbf{R}^p \to Y$ is, for purely algebraic reasons, of form $(t_1, \ldots, t_p) \mapsto (\prod t_j) \cdot x$ for some unique $x \in X$. From Corollary 1.3, we therefore get that f can be uniquely written

$$f(t) = \sum_{p<k} t^p \cdot y_p + h(t)$$

with h of order $\geq k$. Now apply Corollary 2.9 to h.

Theorem 2.11 *Any smooth* $f : \mathbf{R}^n \to X$ *of order* $\geq k$ *can be written (using standard conventions on multi-indices* α*)*

$$f(t) = \sum_{|\alpha|=k} \underline{t}^\alpha \cdot h_\alpha(\underline{t}) \tag{17}$$

for suitable smooth $h_\alpha : \mathbf{R}^n \to X$ *(not uniquely determined in general).*

Proof. This we shall prove by induction on n, combining the $n-1$ result for X with the $n=1$ result for $C^\infty(\mathbf{R}^{n-1}, X)$ (which we have by Corollary 2.9). Denote (temporarily) the linear subspace of $C^\infty(\mathbf{R}^n, X)$ consisting of functions of the form (17) by Eff_k ("functions that are of *effective* order $\geq k$"); it is clearly a subspace of $Ord_k(\mathbf{R}^n, X)$. For $n = 1$, the converse inclusion

$$Ord_k \subseteq \mathrm{Eff}_k(\mathbf{R}^n, X) \tag{18}$$

follows from Corollary 2.9. Assume we have the inclusion (18) for a given n, and for all convenient vector spaces X. Suppose now $f : \mathbf{R}^{n+1} \to X$ is smooth and of order $\geq k$. By exponential adjointness, f corresponds to a smooth

$$\hat{f} : \mathbf{R} \to C^\infty(\mathbf{R}^n, X)$$

to which we now apply Proposition 2.10 to obtain

$$\hat{f}(t) = \sum_{p<k} t^p \cdot y_p + t^k \cdot \hat{g}(t) \tag{19}$$

with $y_p \in C^\infty(\mathbf{R}^n, X)$ and $\hat{g}$ a smooth curve $\mathbf{R} \to C^\infty(\mathbf{R}^n, X)$. Now (20) is equivalent, under exponential adjointness, to

$$f(t, \underline{s}) = \sum_{p<k} t^p \cdot y_p(\underline{s}) + t^k \cdot g(t, \underline{s}).$$

The last term clearly $\in \mathrm{Eff}_k(\mathbf{R}^{n+1}, X)$ due to the factor t^k. For each term $t^p \cdot y_p(\underline{s})$, write by Corollary 1.3 and by induction assumption

$$y_p(\underline{s}) = h_p(\underline{s}) + r_p(\underline{s})$$

with h_p polynomial of degree $< k - p$, and with $r_p \in \mathrm{Eff}_{k-p}(\mathbf{R}^n, X)$. Then $t^p \cdot r_p(\underline{s})$ (as a function of $(t, \underline{s})$) belongs to $\mathrm{Eff}_k(\mathbf{R}^{n+1}, X)$, and $t^p \cdot h_p(\underline{s})$ is polynomial of degree $< k$. Summing over p, and using $Ord_k \cap Poly_{<k} = \{0\}$, we conclude from $f(t, \underline{s}) \in Ord_k$ that $\sum t^p \cdot h_p(\underline{s}) = 0$, so

$$f(t, \underline{s}) = \sum_{p<k} t^p \cdot r_p(\underline{s}) + t^k \cdot g(t, \underline{s}),$$

which belongs to $\mathrm{Eff}_k(\mathbf{R}^{n+1}, X)$. This proves the theorem.

The explicit form of the remainder in Taylor expansions which this theorem provides, yields, by making explicit the polynomial part of such expansion, the following reformulation:

Theorem 2.12 *Let $f : \mathbf{R}^n \to X$ be smooth. Let $k \geq 0$ be an integer. There exist smooth functions $g_\alpha : \mathbf{R}^n \to X$ such that*

$$f(\underline{t}) = \sum_{|\alpha|<k} \frac{1}{|\alpha|!} \frac{\partial^{|\alpha|} f}{\partial \underline{t}^\alpha}(0)\underline{t}^\alpha + \sum_{|\alpha|=k} \underline{t}^\alpha \cdot g_\alpha(\underline{t}). \tag{20}$$

Here, $\partial^{|\alpha|} f/\partial \underline{t}^\alpha$ are iterated partial derivatives $\partial f/\partial t_i$, which in turn can be defined the standard way from df, or, alternatively, can be defined as f_i' where $f_i : \mathbf{R} \to C^\infty(\mathbf{R}^{n-1}, X)$ is one of the exponential adjoints of f.

Proof. The difference of f with the $\sum_{|\alpha|<k}$ sum is proved to be of order $\geq k$, by the technique used in the proof of Theorem 1.2 (noting that

$$\partial(\phi \circ f)/\partial t_i = \phi \circ (\partial f/\partial t_i)$$

for ϕ smooth linear), and utilizing the validity of (20) in the case $X = \mathbf{R}$. So $f - \sum_{|\alpha|}$ is of the required form, by Theorem 2.11.

We end by giving an alternative description of the notion 'order $\geq k$' (Definition 1.1) for an arbitrary smooth $f : X \to Y$.

Theorem 2.13 *Let $f : X \to Y$ be smooth. Then f is of order $\geq k$ iff there exists a smooth $G : \mathbf{R} \times X \to Y$ such that*

$$f(t \cdot x) = t^k \cdot G(t, x) \; \forall (t, x) \in \mathbf{R} \times X.$$

Proof. The implication $\Leftarrow$ is trivial. Assume next that f is of order $\geq k$. Let $F : \mathbf{R} \to C^\infty(X, Y)$ be defined by $F(t)(x) = f(t \cdot x)$. Write by Proposition 2.10, applied to $C^\infty(X, Y)$,

$$F(t) = \sum_{p<k} t^p \cdot y_p + t^k \cdot g(t)$$

with $y_p \in C^\infty(X, Y)$ and $g : \mathbf{R} \to C^\infty(X, Y)$ smooth. For any fixed $x \in X$ and $\phi \in Y'$, we thus have

$$\phi(F(t)(x)) = \sum_{p<k} t^p \cdot \phi(y_p(x)) + t^k \cdot \phi(g(x)).$$

But the left hand side is, as a function $\mathbf{R} \to \mathbf{R}$ of t, equal to $\phi(f(t \cdot x))$ and thus by assumption of order $\geq k$. Standard finite dimensional calculus thus tells us that $\phi(y_p(x)) = 0$. Since x, ϕ were arbitrary, and Y' separates points, $y_p = 0$, so (2) yields

$$F(t) = t^k \cdot g(t) \in C^\infty(X, Y)$$

or

$$f(t \cdot x) = t^k \cdot G(t, x),$$

where $G : \mathbf{R} \times X \to Y$ is the exponential adjoint of $g : \mathbf{R} \to C^\infty(X, Y)$ and thus is smooth.

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