arXiv:math/0407242v1 [math.CT] 14 Jul 2004

Categorical distribution theory; heat equation

A. Kock G.E. Reyes

Introduction

The simplest notion by which a theory of function spaces may be formulated is that of cartesian closed categories. To realize this concretely for spaces of smooth ($= C^\infty$) functions, several notions of diffeological spaces and convenient vector spaces have been developed, besides the whole body of topos theory. Topos theory in particular provides for toposes containing the category of smooth manifolds as full subcategory. In fact, Grothendieck's "Smooth Topos" is closely related to the category of diffeological spaces. The special features of Convenient Vector Spaces were utilized by the present authors back in the 1980's for a more elaborate topos , cf. [12], [14]. The topos there (Dubuc's "Cahiers Topos") in fact accomodates synthetic differential geometry. The present work is a continuation of our work from the 80's, and is motivated by the desire to have a synthetic theory of some of the fundamental partial differential equations, like the heat equation. This forced us to sort out how distribution theory (in the sense of L. Schwartz) relates to convenient vector space theory and to the Cahiers Topos. Note that for the heat equation, distributions of compact support will not suffice (distributions of compact support are easier to deal with in categorical terms, as we did in our paper on the wave equation, [16]).

In particular, we study smoothness with respect to time of solutions of the heat equation[1]. These solutions model evolution through time of a *heat distribution*. A heat distribution is an *extensive* quantity and does not necessarily have a density *function*, which is an *intensive* quantity. The most important of all distributions, the point- or Dirac- distributions, do not. For the heat equation, it is well known that the evolution through time of any distribution leads 'instantaneously' (i.e., after any *positive* lapse of time $t > 0$)

[1] on the unlimited line; we did not work out the details for higher dimensional Euclidean spaces, let alone Riemannian manifolds.

to distributions that do have smooth density functions. Indeed, the evolution through time of the Dirac distribution $\delta(0)$ is given by the map ("heat kernel", "fundamental solution")

$$K : \mathbf{R}_{\geq 0} \to \mathcal{D}'(\mathbf{R}) \tag{1}$$

defined by cases by the classical formula

$$K(t) = \begin{cases} e^{-x^2/4t}/\sqrt{4\pi t} & \text{if } t > 0 \\ \delta(0) & \text{if } t = 0 \end{cases} ; \tag{2}$$

here $\mathcal{D}'(\mathbf{R})$ denotes a suitable space of distributions (in the sense of [23], [22]); notice that in the first clause we are identifying distributions with their density functions (when such density functions exist).

The fundamental mathematical object given in (2) presents a challenge to the synthetic kind of reasoning in differential geometry, where a basic tenet is "everything is smooth"; therefore, definition by cases, as in (2), has a dubious status. It was this challenge that motivated the present study; more precisely, we wanted to present a model of Synthetic Differential Geometry where the map (1) does exist, and satisfies the heat equation, (as well as (2)). We shall in fact construct such K in the Cahiers topos [3].

This construction leads to some smoothness questions that have a purely classical formulation, see Section 4 below.

One may see another lack of smoothness in (2), namely "$\delta(0)$ is not smooth"; but this "lack of smoothness" is completely spurious, when one firmly stays in the space of distributions and their intrinsic "diffeology", in particular avoiding to view distributions as generalized functions. We describe in Section 2 the distribution theory that is adequate for the purpose. In fact, as will be seen in Section 9, this theory is forced on us by synthetic considerations in the Cahiers topos.

We want to thank Henrik Stetkær for useful conversations on the topic of distributions.

1 Diffeological spaces and convenient vector spaces

We collect some notions and facts. Some references are collected at the end of the section.

A *diffeological space* is a set X equipped with a collection of *smooth plots*, a plot p being a map from (the underlying set of) an open set U of some $\mathbf{R}^n$ into X, $p : U \to X$; the collection should satisfy certain stability properties. These properties are best summarized by considering the following site $\underline{mf}$: its objects are open subsets of $\mathbf{R}^n$, the maps are smooth maps between such sets; a covering is a jointly surjective family of local diffeomorphisms. (This site is a site of definition of the "Smooth Topos" of Grothendieck et al., [1] p. 318; and is one of the first examples of what they call a "Gros Topos".) Any set X gives rise to a presheaf $c(X)$ on this site, namely $c(X)(U) := Hom_{sets}(U, X)$. A diffeological structure on the set X is a subsheaf P of the presheaf $c(X)$, the elements of $P(U)$ are called the *smooth* U-plots on X. A set theoretic map $f : X \to X'$ between diffeological spaces is called *(plot-) smooth* if $f \circ p$ is a smooth plot on X' whenever p is a smooth plot on X.

Any smooth manifold M carries a canonical diffeology, namely with $P(U)$ being the set of smooth maps $U \to M$. We have full inclusions of categories: smooth manifolds into diffeological spaces into the smooth topos, (= the topos of sheaves on the site $\underline{mf}$),

$$\underline{Mf} \subseteq \underline{Diff} \subseteq \underline{sh}(\underline{mf}).$$

If X is a diffeological space, and $H \subseteq X$ a subset, there is an induced diffeology on H, namely by declaring $U \to H$ to be a smooth plot iff it is a smooth plot viewed as a map into X. In particular the non-negative reals (=the closed half line) $\mathbf{R}_{\geq 0} \subseteq \mathbf{R}$ will be considered a diffeologicaal space with the diffeology induced by that of $\mathbf{R}$.

Let $\mathbf{R}_{>0}$ denote the open half line of positive reals. A smooth function $f : \mathbf{R}_{>0} \to \mathbf{R}$ is called *square-smooth* if $f(x^2)$ is of the form $g(x)$ for a smooth function $g : \mathbf{R} \to \mathbf{R}$ (necessarily unique).

Note that the square root function is smooth on $\mathbf{R}_{>0}$, but not square smooth, since $\sqrt{x^2} = |x|$, which does not extend smoothly to the whole line. Note also that if f is square smooth, then it extends (uniquely) to a continuous function on the closed half line $\mathbf{R}_{\geq 0}$, by putting $f(0) = g(0)$.

Proposition 1.1 *If $f : \mathbf{R}_{>0} \to \mathbf{R}$ is square smooth, then so is f'.*

Proof. Let $f(x^2) = g(x)$, with g smooth. Then clearly g is an even function, so $g'(0) = 0$. Therefore, $g'(x) = x \cdot h(x)$ for a unique smooth function h. Also

we note that since g is even, g' is odd. For $t > 0$, we have $f(t) = g(t^{1/2})$, and so for $t > 0$,

$$f'(t) = \frac{1}{2}\, g'(t^{1/2}) \cdot t^{-1/2}.$$

For $x \neq 0$, we therefore have (using $\sqrt{(}x^2) = |x|$)

$$f'(x^2) = \frac{1}{2} g'(|x|) \cdot |x|^{-1},$$

but since g' is odd, $g'(|x|) \cdot |x|^{-1} = g'(x) \cdot x^{-1}$. Thus, for $x \neq 0$,

$$f'(x^2) = \frac{1}{2}\, g'(x) \cdot x^{-1} = \frac{1}{2}\, x \cdot h(x) \cdot x^{-1}$$

which extends to the smooth function $1/2\ h(x)$, defined on the whole line. This proves that f' is square smooth.

A smooth function $f : \mathbf{R}_{>0} \to \mathbf{R}$ is called *Seeley-smooth* (after [24]) if all the higher derivatives $f^{(k)} : \mathbf{R}_{>0} \to \mathbf{R}$ have finite limits as $t \to 0+$, i.e. if each $f^{(k)}$ extends to a continuous function defined on the closed half line $\mathbf{R}_{\geq 0}$. A Corollary of the Proposition is then

Proposition 1.2 *If a function $f : \mathbf{R}_{>0} \to \mathbf{R}$ is square-smooth, it is Seeley smooth.*

Proof. We already observed that a square smooth $f : \mathbf{R}_{>0} \to \mathbf{R}$ extends continuously to $\mathbf{R}_{\geq 0}$. But since by Proposition 1.1, all higher derivatives of f are also square smooth, they extend continuously as well.

Theorem 1.1 *For a function $f : \mathbf{R}_{\geq 0} \to \mathbf{R}$, the following conditions are equivalent*

1) f is plot smooth;
2) the restriction of f to $\mathbf{R}_{>0}$ is square smooth;
3) the restriction of f to $\mathbf{R}_{>0}$ is Seeley smooth;
4) f extends to a smooth function defined on the whole of $\mathbf{R}$.

A function $\mathbf{R}_{\geq 0} \to \mathbf{R}$ satisfying these conditions, we will simply call *smooth*.

Proof. 1) implies 2), since the function x^2 is one of the plots we may use; 2) implies 3), by Proposition 1.2; 3) implies 4) by Seeley's Theorem, [24]

(alternatively, by Borel's extension theorem for formal power series, see e.g. [18] p. 18); and clearly, the restriction of a global smooth function to $\mathbf{R}_{\geq 0}$ (or to any other subset of $\mathbf{R}$, with its induced diffeology) is plot smooth (= diffeologically smooth); so 4) implies 1). (An alternative proof of 1) $\Rightarrow$ 4) follows from Whitney's theorem on even functions, cf. [20].)

Note that it follows that if f is Seeley smooth, then the assumed limit $f^{(k)}(t)$ as $t \to 0+$ is the k'th derivative at 0 of (any smooth extension of) f.

The category of diffeological spaces $\underline{Diff}$ is cartesian closed (in fact, it is a concrete quasi-topos). Thus, if X and Y are diffeological spaces, Y^X has for its underlying set the set of smooth maps $X \to Y$; and a map $U \to Y^X$ is declared to be a smooth plot if its transpose $U \times X \to Y$ is smooth. The inclusion into the smooth topos preserves the cartesian closed structure.

For any smooth manifold M, we have in particular a diffeology on $C^\infty(M) = \mathbf{R}^M$, namely a map $g : U \to C^\infty(M)$ is declared to be a smooth plot iff its transpose $U \times M \to \mathbf{R}$ is smooth.

Topological vector spaces V carry a canonical diffeology: a plot $f : U \to V$ is declared to be smooth if for every continuous linear functional $\phi : V \to \mathbf{R}$, $\phi \circ f : U \to \mathbf{R}$ is smooth in the standard sense of multivariable calculus.

A *convenient vector space* is a toplogical vector space with certain properties, cf. [6] 2.6.3; one of these is: if $\phi : V \to \mathbf{R}$ is a linear functional, which is smooth with respect to the diffeological structures on V and $\mathbf{R}$, then ϕ is continuous. (So the set of continous linear functionals is closed under a certain obvious Galois correspondence between linear functionals $V \to \mathbf{R}$, on the one side, and plots $\mathbf{R}^n \supseteq U \to V$ on the other.) – We use "CVS" as a shorthand for the phrase "convenient vector space".

Note: Besides the category of convenient vector spaces as a full subcategory of the category of topological vector spaces, we shall have occasion to consider another much larger category $\underline{Con}^\infty$ of convenient vector spaces; it has the same objects, but with *all* smooth maps in between them, not just the smooth *linear* ones. The category $\underline{Con}^\infty$ is a full subcategory of the category $\underline{Diff}$ of diffeological spaces.

For convenient vector spaces, a map $f : X \to Y$ is plot smooth iff it is *scalarwise* smooth, meaning that $\phi \circ f : X \to \mathbf{R}$ is smooth for any $\phi \in Y'$ (where Y' is the set of continuous (=smooth) linear functionals $Y \to \mathbf{R}$).

Convenient vector spaces X have the following completeness property: given a smooth curve $g : U \to X$ (i.e. a smooth plot, where U an open

interval in $\mathbf{R}$), there is a unique function $g' : U \to X$ which is derivative of g in the scalarwise sense that $(\phi \circ g)' = \phi \circ g'$ for all $\phi \in X'$; and this g' is itself smooth. – More generally, if $U \subseteq \mathbf{R}^n$ is open, and $g : U \to X$ is a smooth plot, then partial derivatives g^α of g exist, in the scalarwise sense; and they are smooth (in particular, they are continuous). Here α is a multi-index; and to say that g^α is an iterated partial derivative of g, in the *scalarwise* sense, is to say: for each $\phi \in X'$, $\phi \circ g$ has an αth iterated derivative, and $(\phi \circ g)^\alpha = \phi \circ g^\alpha$.

Conversely, if $g : U \to X$ has the property that scalarwise iterated partial derivatives g^α exist, and are scalarwise continuous, then for each $\phi \in X'$, $\phi \circ g$ has iterated partial derivatives, and they are continuous, since the g^α were assumed to be so; so $\phi \circ g$ is smooth, and therefore g itself is smooth.

For $i : X \to Y$ be a smooth linear map between convenient vector spaces. Then i preserves differentiation of smooth plots $U \to X$, in an obvious sense. For instance, if $f : U \to X$ is a smooth curve, i.e. $U \subseteq \mathbf{R}$ an open interval, then for any $t_0 \in U$,

$$(i \circ f)'(t_0) = i(f'(t_0)).$$

For, it suffices to test this with the elements $\psi \in Y'$. If $\psi \in Y'$, then $\psi \circ i \in X'$ since i is smooth and linear, and the result then follows by definition of being a scalarwise derivative in X.

We don't know at present whether generally scalarwise smooth curves $\mathbf{R}_{\geq 0} \to X$ similarly have "scalarwise derivatives" in the endpoint 0 (unless X is $\mathbf{R}$, say, where the result follows from Theorem 1.1). This prompts us to make a definition.

Definition 1.1 *Call a map $g : \mathbf{R}_{\geq 0} \to X$* strongly smooth *if for each natural number n, there exists a map $g^{(n)} : \mathbf{R}_{\geq 0} \to X$ such that for each $\phi \in X'$, $\phi \circ g$ is n times differentiable with $(\phi \circ g)^{(n)} = \phi \circ g^{(n)}$.*

If $g : \mathbf{R}_{\geq 0} \to X$ is strongly smooth (with X a CVS), it is scalarwise smooth, in the sense that $\phi \circ g : \mathbf{R}_{\geq 0} \to \mathbf{R}$ satisfies the condition (4) of Theorem 1.1, for every $\phi \in X'$. Hence, by the Theorem, $\phi \circ g$ is also smooth for every $\phi \in X'$, and by definition of the diffeology on the CVS X, g itself is smooth.

So "strongly smooth implies smooth", for maps $\mathbf{R}_{\geq 0} \to X$.

(Another aspect of the completeness of convenient vector spaces is: if U is an open interval, and $u_0 \in U$, there is a unique smooth primitive G

($G' = g$) of g, with $G(u_0) = 0$. This is the basis for constructing "Hadamard remainders" with values in a CVS, and hence for the comparisons of the present Section 5.)

Pointers to the literature: Convenient Vector Spaces were introduced by Frölicher and Kriegl, an exposition is in [6]; diffeological spaces (cf. [25]) seem to have been invented and re-invented with small variations and with different names several times, because they seem not to be really admitted into mainstream functional analysis. One early reference is Chen's [2], were a variation of the theme, under the name "Differential Space" is introduced. Convenient Vector Spaces are put into the context of diffeological spaces in [19]. A recent account is given in [26], where also a comparison with convenient vector spaces is presented.

The category $\underline{C}^\infty$ of *smooth spaces*, [6] 1.4.1, is a full subcategory of the category of diffeological spaces, but it does not enter directly in our exposition.

2 The basic vector spaces of distribution theory; test plots

Let M be a smooth (paracompact) manifold M (we shall here be interested in $\mathbf{R}^m$, only). Distribution theory starts out with the vector space $C^\infty(M)$ of smooth real valued functions on M, and the linear subspace $\mathcal{D}(M) \subseteq C^\infty(M)$ consisting of functions with compact support ($\mathcal{D}(M)$ is the "space of *test functions*). The topology relevant for distribution theory is described (in terms of convergence of sequences) in [22], p. 79 and 108, respectively. Note that the topology on $\mathcal{D}(M)$ is finer than the one induced from the topology on $C^\infty(M)$. The sheaf semantics which we shall consider in Section 7 will justify the choice of these topologies.

We shall describe the diffeological structure, arising from the topology on $\mathcal{D}(M)$, and utilize the fact ([6], Remark 3.5) that it is a convenient vector space.

We cover M by an increasing sequence K_b of compact subsets, $M = \cup K_b$; the notions that we now describe are independent of the choice of these K_b. For $M = \mathbf{R}^n$, we would typically take $K_b = \{x \in \mathbf{R}^n \mid |x| \leq b\}$, $b \in \mathbf{N}$.

Consider a smooth map $f : U \times M \to \mathbf{R}$, where U is an open subset of some $\mathbf{R}^n$. We say that it is of *uniformly bounded support* if there exists b so

that

$$f(u,x) = 0 \text{ for all } u \in U \text{ and all } x \text{ with } x \notin K_b$$

We say that f is *locally* of uniformly bounded support ("l.u.b.s.") if U can be covered by open subsets U_i such that for each i, the restriction of f to $U_i \times M$ is of uniformly bounded support. (We may use the phrase "f is l.u.b.s., *locally in the variable* $u \in U$") - Equivalently, we say f is of uniform bounded support *at* $u \in U$ if there is an open neighbourhood U' around u such that the restriction of f to $U' \times M$ is of uniformly bounded support; and f is l.u.b.s. if it for each u is of uniformly bounded support at u. (For yet another description of the notion, see Lemma 7.1 below.)

We let $\hat{f}$ denote the transpose of f, so $\hat{f} : U \to C^\infty(M)$.

Theorem 2.1 *Let $f : U \times M \to \mathbf{R}$ be smooth, and pointwise of bounded support (so that $\hat{f}$ factors through $\mathcal{D}(M)$). Then t.f.a.e.:*
1) f is locally of uniformly bounded support
2) $\hat{f} : U \to \mathcal{D}(M)$ is continuous.

We may use the term *test plot* for functions f satisfying the conditions of the Proposition. Pointwise, they are test *functions* in the sense of distribution theory.

Proof of the Theorem. We first prove that 1) implies 2). Since the question is local in U, we may assume that f is of uniformly bounded support, i.e. there exists a compact $K \subseteq M$ so that $f(t,x) = 0$ for $x \notin K$ and all t. The same K applies then to all the iterated partial derivatives f_α of f in the M-directions (α denoting some multi-index). So f and all the f_α factor through $\mathcal{D}_K$, the subset of $C^\infty(M)$ of functions vanishing outside K. Now to say that $\hat{f} : U \to \mathcal{D}_K$ is continuous is by definition of the topology on $\mathcal{D}_K$ equivalent to saying that for each α, $(f_\alpha)\hat{}$ is continuous as a map into $\mathbf{R}^K$, the space of continuous maps $K \to \mathbf{R}$, with the topology of uniform convergence. This topology is the categorical exponent (= compact open topology) (cf. [8] Ch. 7 Thm. 11), which implies that $(f_\alpha)\hat{} : U \to \mathbf{R}^K$ is continuous iff $f_\alpha : U \times K \to \mathbf{R}$ is continuous, iff $f_\alpha : U \times M \to \mathbf{R}$ is continuous. But f_α is indeed continuous, by the smoothness assumption on f. So $\hat{f} : U \to \mathcal{D}(M)$ is continuous.

For proving that 2) implies 1), we prove that if not 1), then not 2), i.e. we consider a function $f : U \times M \to \mathbf{R}$ which is smooth and of pointwise bounded support, but not l.u.b.s. Then there is a $t_0 \in U$ and a sequence

$t_k \to t_0$, as well as a sequence $x_k \in M \setminus K_k$ with $f(t_k, x_k) \neq 0$, denote this number c_k. Let N be a number so that the support of $f(t_0, -)$ is contained in K_N. We consider the (non-linear) functional $T : \mathcal{D}(M) \to \mathbf{R}$ given by

$$g \mapsto \sum_{n=N}^{\infty} c_n^{-2} g(x_n)^2.$$

Note that for g of compact support, this sum is finite, since the x_n's "tend to infinity". Also, the functional $\mathcal{D}(M) \to \mathbf{R}$ is continuous; for the topology on $\mathcal{D}(M)$ is the inductive limit of the topology $\mathcal{D}(K_k)$, and the restriction of T to this subspace equals a *finite* algebraic combination of the Dirac distributions. Now it is easy to see that T takes $f(t_0, -)$ to 0, by the choice of N, whereas T applied to $f(t_k, -)$ for $k > N$ yields a sum of non-negative terms, one of which has value 1, namely the one with index k, which is $c_k^{-2} f(t_k, x_k)^2 = 1$. So $T \circ \hat{f}$ is not continuous, hence $\hat{f}$ is not continuous.

This proves the Theorem.

It has the following Corollary:

Theorem 2.2 *Let $f : U \times M \to \mathbf{R}$ be smooth and of pointwise bounded support (U an open subset of some $\mathbf{R}^n$). Then t.f.a.e.:*

1) f is locally of uniformly bounded support

2) $\hat{f} : U \to \mathcal{D}(M)$ is smooth.

Recall that assertion 2) means "in the scalarwise sense", i.e. $\phi \circ \hat{f}$ is smooth for any continuous linear functional, i.e. for any distribution ϕ.

Proof. The implication 2) implies 1) is a consequence of Theorem 2.1, since smoothness implies continuity. Conversely, assume 1), i.e. assume f is smooth and l.u.b.s. Then we also have that $\partial^\alpha f / \partial t^\alpha$ is smooth (iterated partial derivative in the U-directions, α a multi-index) and l.u.b.s., and so its transpose is a continuous maps $U \to \mathcal{D}(M)$, by Theorem 2.1; it serves as scalarwise iterated partial derivative. (This is an entirely classical statement; we could not find an explicit reference, so we sketch a proof. The continuous linear functionals that define what "scalarwise" means are by definition the distributions on M. For the case where M and U both are $\mathbf{R}$, it is thus the assertion that for any distribution T, if $f(t, s) : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is smooth, and, locally in the variable t, of uniformly bounded support, then $t \mapsto T([s \mapsto f(t, s)])$ has a t-derivative which is given by $t \mapsto T([s \mapsto \partial f(t, s)/\partial t])$, i.e. "one can differentiate under the distribution sign". Now since the desired

conclusion is of local nature in t, we may w.l.o.g. assume that f is of uniformly bounded support, and then we may modify T so as to have compact support also. Then T may be represented by a finite sum of "derivatives of continuous functions" (cf. [23] Thm. 26), so the assertion of "differentiating under the distribution sign" becomes essentially the assertion that you may differentiate under the integration sign, which is possible due to the compactness of the support.)

The standard vector space of distributions $\mathcal{D}'(M)$ is, in diffeological terms, the linear subspace of the diffeological space $\mathbf{R}^{\mathcal{D}(M)}$ consisting of the *linear* smooth maps $\mathcal{D}(M) \to \mathbf{R}$. A map $U \to \mathcal{D}'(M)$ is smooth iff it is smooth as a map into $\mathbf{R}^{\mathcal{D}(M)}$; this defines a diffeology on $\mathcal{D}'(M)$. With this diffeology, $\mathcal{D}'(M)$, too, is convenient.

The diffeology/convenient vector space structure on $\mathcal{D}(M)$ corresponds to its standard locally convex topology, so that a linear functional $\mathcal{D}(M) \to \mathbf{R}$ is diffeological iff it is continuous. So the vector space of distributions $\mathcal{D}'(M)$ (as an abstract vector space) is the same in both contexts.

We have

Theorem 2.3 *The convenient vector space $\mathcal{D}(M)$ is reflexive in the CVS sense: the canonical $\mathcal{D}(M) \to \mathcal{D}''(M)$ is an isomorphism.*

We presume that this result is well known among experts, but it is not explicitly stated in [6], say. It is known that, as a locally convex topological vector space, $\mathcal{D}(M)$ is reflexive, with respect to the so called strong topology on dual spaces, cf. [23] Theorem XIV. Now [6] has a general Theorem, comparing reflexivity in various categories of vector spaces (locally convex, convenient, bornological, ...), namely Theorem 5.4.6. By this Theorem, convenient reflexivity follows from strong reflexivity, provide that the strong dual (in our case $\mathcal{D}'(M)$ with its strong topology) is furthermore bornological. And this is known to be so, cf. e.g. [7] Example 3.16.2.

3 Functions as distributions

Any sufficiently nice function $f : \mathbf{R}^n \to \mathbf{R}$ gives rise to a distribution $i(f) \in \mathcal{D}'(\mathbf{R}^n)$ in the standard way "by integration over $\mathbf{R}^n$"

$$< i(f), \phi >:= \int_{\mathbf{R}^n} f(s) \cdot \phi(s)\, ds.$$

This also applies if $\mathbf{R}^n$ is replaced by another smooth manifold M equipped with a suitable measure. For simplicity of notation, we write M for $\mathbf{R}^n$ in the following. – All smooth functions $f : M \to \mathbf{R}$ are "sufficiently nice"; so we get a map (obviously linear)

$$i : C^\infty(M) \to \mathcal{D}'(M). \tag{3}$$

It is also easy to see that this map is injective.

Theorem 3.1 *The map i is smooth.*

Proof. Let $g : V \to C^\infty(M)$ be smooth, (V an open subset of some $\mathbf{R}^n$), we have to see that $i \circ g : V \to \mathcal{D}'(M)$ is smooth, which in turn means that its transpose

$$(i \circ g)\hat{} : V \times \mathcal{D}(M) \to \mathbf{R}$$

is smooth. So consider a smooth plot $U \to V \times \mathcal{D}(M)$, given by a pair of smooth maps $h : U \to V$ and $\hat{\Phi} : U \to \mathcal{D}(M)$. Here U is again an open subset of some $\mathbf{R}^k$. Let us write $\hat{F}$ for $g \circ h : U \to C^\infty(M)$. It is transpose of a map $F : U \times M \to \mathbf{R}$. Also, let us write Φ for the transpose of $\hat{\Phi}$; thus Φ is a map

$$\Phi : U \times M \to \mathbf{R}$$

which is locally (in U) of uniformly bounded support, by Theorem 2.2. We have to see that $(i \circ g)\hat{} \circ < h, \Phi >$ is smooth (in the usual sense). By unravelling the transpositions, one can easily check that

$$(i \circ g)\hat{} \circ < h, \Phi > (t) =< i(F(t, -), \Phi(t, -) >$$

The conclusion of the Theorem is thus the assertion that the composite map $U \to \mathbf{R}$ given by

$$t \mapsto \int_M F(t, s) \cdot \Phi(t, s)\, ds \tag{4}$$

is smooth (in the standard sense of finite dimensional calculus). To prove smoothness at $t_0 \in U$, we may find a neigbourhood U' of t_0 and a b such that

$$\Phi(t, s) = 0 \text{ if } t \in U' \text{ and } s \notin K_b,$$

because Φ is l.u.b.s. We thus have, for any $t \in U'$, that the expression in (4) is $\int_{K_b} F(t, s) \cdot \Phi(t, s)\, ds$, but since K_b is compact, differentiation and other limits in the variable t may be taken inside the integration sign.

Since $i : C^\infty(M) \to \mathcal{D}'(M)$ is smooth and linear, it preserves differentiation. In particular, if $f : U \to C^\infty(M)$ is a smooth curve, and $t_0 \in U$, we have that $(i \circ f)'(t_0) = i(f'(t_0))$. However, f' is explicitly calculated in terms of the partial derivative of the transpose $\hat{f} : U \times M \to \mathbf{R}$, namely as the function $s \mapsto \partial f(t,s)/\partial t \mid_{(t_0,s)}$. This is the reason that ordinary (evolution-) differential equations for curves $f : U \to \mathcal{D}'(M)$ manifest themselves as *partial* differential equations, as soon as the values of f are distributions represented by smooth functions.

4 Smoothness of heat kernel

We consider the heat equation on the line,

$$\partial f/\partial t(t,x) = \partial^2 f/\partial x^2.$$

Recall that the classical distribution solution of this equation, having $\delta(0)$ as initial distribution, is the map

$$K : \mathbf{R}_{\geq 0} \to \mathcal{D}'(\mathbf{R})$$

whose value at $t \geq 0$ is the distribution $< K(t), _ >$ given on a test function ϕ by

$$< K(t), \phi >= \begin{cases} \int_\infty^\infty e^{-s^2/4t}/\sqrt{4\pi t}\ \phi(s)\ ds & \text{if } t > 0 \\ \phi(0) & \text{if } t = 0 \end{cases} \tag{5}$$

The present Section is devoted to proving the strong smoothness (Definition 1.1), and hence also the diffeological smoothness, of K.

Theorem 4.1 *The function $K : \mathbf{R}_{\geq 0} \to \mathcal{D}'(\mathbf{R})$ is strongly smooth.*

Proof. We have to produce for each n a map $K^{(n)} : \mathbf{R}_{\geq 0} \to \mathcal{D}'(\mathbf{R})$ which will serve as the n'th scalarwise derivative of K. The map $\Delta^n \circ K$ will do. For, consider a smooth linear $\rho : \mathcal{D}'(\mathbf{R}) \to \mathbf{R}$. So $\rho \in \mathcal{D}''(\mathbf{R})$, but by CVS-reflexivity of $\mathcal{D}(\mathbf{R})$ (Theorem 2.3), ρ is of the form $\rho(T) =< T, \phi >$ for a unique test function $\phi \in \mathcal{D}(\mathbf{R})$.

So it suffices to prove, for each test function ϕ, that $< (\Delta^n \circ K)(t), \phi >$, as a function of $t \in \mathbf{R}_{\geq 0}$, is the n'th derivative of $< K(t), \phi >$. Now,

$$< (\Delta^n \circ K)(t), \phi >=< K(t), \phi^{(2n)} >$$

(recalling $\Delta = (-)''$, and the differentiation of distributions); so this reduces the problem to proving the following:

Proposition 4.1 *Let ϕ be a smooth function of compact support and let $\Phi : \mathbf{R}_{\geq 0} \to \mathbf{R}$ be the function defined by*

$$\Phi(t) =< K(t), \phi > .$$

Then the function Φ is smooth and, furthermore, for all $t \geq 0$, $\Phi^{(n)}(t) = \phi^{(2n)}(t)$.

For $t > 0$, this is well known: for all $t > 0$, the n'th derivative of Φ exists at t, and

$$\Phi^{(n)}(t) =< K(t), \phi^{(2n)} >, \tag{6}$$

see p. 330 in [22]). Next, we prove

Lemma 4.1 *For every test function $\phi : \mathbf{R} \to \mathbf{R}$*

$$lim_{t \to 0^+}(1/t)[< K(t), \phi > - \phi(0)] = \phi''(0)$$

Proof: We first notice that, by Hadamard's Lemma, $\phi(x) = \phi(0) + x\psi(x)$, for a unique smooth function ψ. (The function ψ goes to 0 when $x \to +\infty$ or $x \to -\infty$ since $\psi(x) = (1/x)[\phi(x) - \phi(0)]$, but does not necessarily have compact support. Similarly for ψ', ψ'', etc.; but this boundedness is enough to make the improper integrals convergent.) Note that $\phi''(0) = 2\psi'(0)$.

We claim that for $t > 0$

$$< K(t), \phi > -\phi(0) = 2t < K(t), \psi' > . \tag{7}$$

In fact, start from the right hand side; we get the limit as N_1 and $N_2 \to \infty$ of

$$2t \int_{-N_1}^{N_2} 1/\sqrt{4\pi t}\; e^{-x^2/4t} \psi'(x) dx$$

which we integrate by parts to get

$$(2t \cdot 1/\sqrt{4\pi t}\; e^{-x^2/4t} \cdot \psi(x))|_{-N_1}^{N_2} + \int_{-N_1}^{N_2} (1/\sqrt{4\pi t}) e^{-x^2/4t} x\psi(x)\; dx.$$

Since ψ is bounded, the first term here tends to 0 as N_1, N_2 tend to ∞, so passing to the limit, and using $x\psi(x) = \phi(x) - \phi(0)$, we get

$$2t < K(t), \psi' >= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} x\psi(x)\; dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} (\phi(x) - \phi(0)\, dx$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} \phi(x)\, dx - \phi(0)$$
$$=< K(t), \phi > - < K(0), \phi > .$$

Now divide by t and let $t \to 0$, recalling that $< K(t), \psi' > \to \psi'(0)$ as $t \to 0$.

Proof of the Proposition: We first show that

$$\Phi^{(n)}(0) = \phi^{(2n)}(0) \tag{8}$$

by induction on n. For $n = 0$ this is by definition of Φ. Assume that it is true for n. Then

$$\begin{aligned} \Phi^{(n+1)}(0) &= lim_{t \to 0^+} (1/t)[\Phi^{(n)}(t) - \Phi^{(n)}(0)] \\ &= lim_{t \to 0^+} (1/t)[< K(t), \phi^{(2n)} > - \phi^{(2n)}(0)] \\ &= (\phi^{(2n)})''(0) \\ &= \phi^{(2(n+1))}(0) \end{aligned}$$

In the passage from the first line to the second we have used the induction hypothesis and (6), whereas to go to the third line from the second we have used Lemma 4.1 with $\phi^{(2n)}$.

We noted already that for a map with domain $\mathbf{R}$ or $\mathbf{R}_{\geq 0}$, and with values in a CVS, strong smoothness implies scalarwise (equivalently, diffeological) smoothness. So Theorem 4.1 has as Corollary:

Theorem 4.2 *The function $K : \mathbf{R}_{\geq 0} \to \mathcal{D}'(\mathbf{R})$ is smooth (in the diffeological sense).*

This Theorem, in turn, translates by passing to the transpose map, the following result, formulated in entirely classical and elementary-calculus terms. (In one of the preliminary versions of the present paper, we gave an elementary proof of it, and hence of Theorem 4.2, without resorting to reflexivity of $\mathcal{D}(M)$.)

Theorem 4.3 *Let $\phi : U \times \mathbf{R} \to \mathbf{R}$ be a test plot (i.e. smooth, and locally (on U) of uniformly bounded support). Then the function $\Phi : U \times \mathbf{R}_{\geq 0} \to \mathbf{R}$ defined by*

$$\Phi(u, t) :=< K(t), \phi(u, -) >$$

is smooth .

5 Ideals and differential operators

Let $x \in \mathbf{R}^n$. By a *differential operator supported at* x, we understand a map $d : C^\infty(\mathbf{R}^n) \to \mathbf{R}$ which is a linear combination of operators $f \mapsto \partial^{|\alpha|} f/\partial t^\alpha(x)$, where α is a multi-index and $t = (t_1, \ldots, t_n)$. (The notion can be defined in a coordinate free way; it is actually the same as a distribution with point-support.) In particular, d is linear.

Any such d defines, because of its explicit form, for each CVS Y a linear $d_Y : C^\infty(\mathbf{R}^n, Y) \to Y$ with the property that for $f : \mathbf{R}^n \to Y$

$$d(\phi \circ f) = \phi(d_Y(f))$$

for all $\phi \in Y'$. The maps d_Y are natural in Y w.r.to smooth *linear* maps:

Proposition 5.1 *If $F : Y \to X$ is a smooth linear map, then for any differential operator d, and any $f \in C^\infty(\mathbf{R}^n, Y)$, $d_X(F \circ f) = F(d_Y(f))$*

Proof. It suffices to test with an arbitrary $\phi \in X'$; by replacing F by $\phi \circ F$, this reduces the problem to the case where the dodomain X is $\mathbf{R}$, and here, the result follows from the very characterization of Y-valued derivatives in "scalarwise" terms.

Let us also note that "partial derivatives are transposable". For simplicity, we state it for functions in two variables s, t only:

Proposition 5.2 *Let $f(s,t) : \mathbf{R}^2 \to Y$ be a smooth function with values in a CVS Y. Then the $\partial f(s,t)/\partial s$ is smooth in s, t, and its transpose is the derivative $(\hat{f})'(s)$ of the transposed function $\hat{f} : \mathbf{R} \to C^\infty(\mathbf{R}, Y)$.*

Proof. The function $(\hat{f})'(s)$ exists and is smooth, and characterized in terms of the smooth linear functionals on $C^\infty(\mathbf{R}, Y)$. But among these are those of the form (for $t \in \mathbf{R}$)

$$C^\infty(\mathbf{R}, Y) \xrightarrow{ev_t} Y \xrightarrow{\psi} \mathbf{R},$$

and these are enough to recognize the transpose of $\hat{f}'(s)$ as $\partial f(s,t)$, for each t.

Let $I \subseteq C^\infty(\mathbf{R}^l)$ be an ideal. For each CVS Y, (in fact for any dualized vector space (Y, Y')) we define two linear subspaces of $C^\infty(\mathbf{R}^l, Y)$, the "weak"

and the "strong" $I(Y)$, denoted $I_w(Y)$ and $I_s(Y)$, respectively. To say that $f : N \to Y$ is in $I_w(Y)$ is to say that for every $\phi \in Y'$, $\phi \circ f \in I$; and to say that $f : N \to Y$ is in $I_s(Y)$ is to say that f may be written

$$f(s) = \sum h_i(s)k_i(s),$$

with the h_i's scalar valued functions belonging to I, and the k_i's smooth Y-valued functions. It is clear that $I_s(Y) \subseteq I_w(Y)$. We are interested in when the converse implication holds.

A main result in [11] (Theorem 2.11) says that this is the case for the ideal $\mathcal{M}^r \subseteq C^\infty(\mathbf{R}^l)$ of functions vanishing to order r at 0. In [14] (Proposition 1), we generalized this to any proper ideal $I \subseteq C^\infty(\mathbf{R}^l)$ which contains an ideal $\mathcal{M}^r$. We call such ideals *Weil ideals*; they are of finite codimension, and the algebra $C^\infty(\mathbf{R}^l)/I$ is a Weil algebra (in the sense of [10] or [18], say); and any Weil algebra arises this way. (Note that a Weil ideal is contained in $\mathcal{M}$, since the only maximal ideal containing $\mathcal{M}^r$ is $\mathcal{M}$. So if $f \in I$, $f(0) = 0$.)

We shall generalize this result further to what we call "semi-Weil ideals" J, and at the same time provide a simpler proof of the result quoted from [14].

If $I \subseteq C^\infty(N)$ is an ideal and if $p : P \to N$ is a smooth map (P and N manifolds), we get an ideal $p^*(I) \subseteq C^\infty(P)$ consisting of functions $f : P \to \mathbf{R}$ which can be written $\sum(h_i \circ p) \cdot k_i$ with the h_i's in I (and the k_i's in $C^\infty(P)$). This is clearly a "transitive" construction, in an evident sense, $q^*(p^*(I)) = (p \circ q)^*(I)$. On the other hand, since $C^\infty(M)$ is a convenient vector space, we may consider $I_s(C^\infty(M)) \subseteq C^\infty(N, C^\infty(M))$. Under the isomorphism $C^\infty(N, C^\infty(M)) \cong C^\infty(N \times M)$ it is clear that $I_s(C^\infty(M))$ corresponds to $p^*(I)$, where $p : N \times M \to N$ denotes the projection.

If I is a Weil ideal $\subseteq C^\infty(\mathbf{R}^l)$, and $p : \mathbf{R}^{l+k} \to \mathbf{R}^l$ the projection, we get by the above procedure an ideal $J = p^*(I)$ in $C^\infty(\mathbf{R}^{l+k})$, and ideals J of this form, we call *semi-Weil ideals*.

The basis of monomials s^α (where α is a multi-index of order $< r$) for $C^\infty(\mathbf{R}^l)/(\mathcal{M}^r)$ gives rise to a dual basis for the linear dual $(C^\infty(\mathbf{R}^l)/\mathcal{M}^r))^*$, and this dual basis consists of differential operators supported at 0,

$$f \mapsto \frac{\partial^\alpha f(0)}{|\alpha|!\partial s^\alpha}.$$

So $f \in \mathcal{M}^r$ iff $\frac{\partial^\alpha f(0)}{s^\alpha} = 0$ for such multi-indices α.

We consider functions $f(s,t) : \mathbf{R}^{l+k} \to Y$, where Y is a CVS; s denotes a variable ranging over $\mathbf{R}^l$ and t a variable ranging over $\mathbf{R}^k$. We then have

Proposition 5.3 *Let $f : \mathbf{R}^{l+k} \to Y$ be a smooth function. Then*

$$f \in (p^*(\mathcal{M}^r))_w(Y)$$

if and only if

$$\frac{\partial^\alpha f(0,t)}{\partial s^\alpha} = 0 \text{ for all } \alpha \text{ with } |\alpha| < r \text{ and all } t.$$

Proof. The Y-valued partial derivatives here are determined scalarwise, i.e. determined by testing with the $\phi \in Y'$, and since these ϕ are linear, the problem immediately reduces to the case of $Y = \mathbf{R}$, i.e. to the assertion $f(s,t) \in p^*(\mathcal{M}^r)$ iff $\frac{\partial^\alpha f(0,t)}{s^\alpha} = 0$ for all α with $|\alpha| < r$ and all t. This is well known (or can be deduced from Theorem 2.11 in [11], by passing to the transpose function $\hat{f} : \mathbf{R}^l \to C^\infty(\mathbf{R}^k)$).

The following is now a Corollary of Theorem 2.11 in [11]:

Proposition 5.4 *For any CVS Y, we have $(p^*(\mathcal{M}^r))_w(Y) = (p^*(\mathcal{M}^r))_s(Y)$.*

Proof. It suffices to prove the inclusion $\subseteq$. If f is in the left hand side, it satisfies the equational conditions of Proposition 5.3, but then its transpose $\hat{f} : \mathbf{R}^l \to C^\infty(\mathbf{R}^k, Y)$ has $\frac{\partial^\alpha \hat{f}(0)}{s^\alpha} = 0$ for all α with $|\alpha| < r$. Now we apply Proposition 5.3 again, this time for the CVS $C^\infty(\mathbf{R}^k, Y)$, and with no p^* involved, and conclude $\hat{f} \in (\mathcal{M}^r)_w(C^\infty(\mathbf{R}^k, Y))$. Then, by the Theorem quoted, $\hat{f} \in (\mathcal{M}^r)_s(C^\infty(\mathbf{R}^k, Y))$ (strong instead of weak), and this in turn implies that $f \in (p^*(\mathcal{M}^r))_s(Y)$, proving the Proposition.

Consider a Weil ideal I i.e. an ideal $I \subseteq C^\infty(\mathbf{R}^l)$ containing some $\mathcal{M}^r$. There is a (finite) basis A for the dual vector space $(C^\infty(\mathbf{R}^l)/\mathcal{M}^r)^*$ consisting of differential operators D^α at 0 (with $\mathcal{M}^r$ the common nullspace of these). Since $(C^\infty(\mathbf{R}^l)/I)^* \subseteq (C^\infty(\mathbf{R}^l)/\mathcal{M}^r)^*$, we may, by suitable change of basis, organize ourselves so that the basis A for $(C^\infty(\mathbf{R}^l)/\mathcal{M}^r)^*$ contains a subset B which is a basis for $(C^\infty(\mathbf{R}^l)/I)^*$. It follows that I is the common null space of the collection B of differential operators.

The dual basis $\hat{A}$ for $C^\infty(\mathbf{R}^l)/\mathcal{M}^r$ consists (modulo $\mathcal{M}^r$) of polynomials h_α of degree $< r$, ($\alpha \in A$). The fact that the bases A and A' are dual implies that for any $f \in C^\infty(\mathbf{R}^l)$,

$$f(s) \equiv \sum_\alpha D^\alpha f \cdot h_\alpha(s),$$

mod $\mathcal{M}^r$ (as functions of $s \in \mathbf{R}^l$). If now $f \in I$, the terms $D^\alpha f$ vanish for $\alpha \in B$. With $A - B$ as index set for the index γ, we therefore have

Proposition 5.5 *Given a Weil-ideal $I \in C^\infty(\mathbf{R}^l)$ containing $\mathcal{M}^r$. There is a finite family of differential operators D^γ and a family of polynomials $h_\gamma(s)$ in $s \in \mathbf{R}^l$ so that for any $f \in I$,*

$$f(s) - \sum_\gamma D^\gamma f \cdot h_\gamma(s) \in \mathcal{M}^r.$$

(If for instance $I = \mathcal{M}^{r-1} \supseteq \mathcal{M}^r$, the h_γ's may be taken to be the monomials s^α, where α ranges over multi-indices with $|\alpha| = r$.)

Because differentiation of functions $\mathbf{R}^l \to Y$ (with Y a CVS) makes sense, and because of the explicit way (in terms of D^γ's) in which functions in I get transformed into functions in $\mathcal{M}^r$, this Proposition immediately extends to functions $\mathbf{R}^l \to Y$; let I and h_γ be as above, and let the D^γ denote the Y-valued differential operators corresponding to the $\mathbf{R}$-valued D^γ's considered.

Proposition 5.6 *For any $f \in I_w(Y)$, the difference*

$$f(s) - \sum_\gamma D^\gamma f \cdot h_\gamma(s)$$

belongs to $\mathcal{M}^r_w(Y)$ (which equals $\mathcal{M}^r_s(Y)$) by the Theorem [11] 2.11 quoted).

Proof. We test with arbitrary $\phi \in Y'$; since ϕ is linear, and since ϕ commutes with differentiation, the result follows by applying the result of the previous Proposition to the smooth function $\phi \circ f$, which is in I by assumption.

Now let J denote the semi-Weil ideal $p^*I \subseteq C^\infty(\mathbf{R}^{l+k})$ given by the Weil ideal $I \subseteq C^\infty(\mathbf{R}^l)$. Then

Proposition 5.7 *Let $f : \mathbf{R}^{l+k} \to \mathbf{R}$ be a function in J. Then*

$$f(s,t) - \sum (D^\gamma f)(t) \cdot h_\gamma(s)$$

is in $p^*(\mathcal{M}^r)$ (where $p : \mathbf{R}^{l+k} \to \mathbf{R}^l$ is the projection).

(Here, s and t denote variables ranging over $\mathbf{R}^l$ and $\mathbf{R}^k$, respectively. The differential operators D^γ operate in the s-variable and then $s = 0$ is substituted, so a function $D^\gamma f$ of t remains, as indicated.)

Proof. We pass to the transpose function $\hat{f} : \mathbf{R}^l \to Y$, where Y is the CVS $C^\infty(\mathbf{R}^k)$. To say $f \in J$ is equivalent to saying $\hat{f} \in I_s(C^\infty(\mathbf{R}^k))$, in particular $\hat{f} \in I_w(C^\infty(\mathbf{R}^k))$, and so Proposition 5.6 may be applied, reducing $\hat{f}$ to $\mathcal{M}^r_s(C^\infty(\mathbf{R}^k)$, which by transposition corresponds to $p^*(\mathcal{M}^r)$. This proves the Proposition.

We generalize this further to the case of functions with values in a CVS Y.

Proposition 5.8 *Let $J \subseteq C^\infty(\mathbf{R}^{l+k})$ be the semi-Weil ideal given by the Weil ideal $I \in C^\infty(\mathbf{R}^l)$. Let $g(s,t) \in J_w(Y)$. Then*

$$g(s,t) - \sum (D^\gamma g)(t) \cdot h_\gamma(s) \tag{9}$$

is in $(p^*(\mathcal{M}^r))_w(Y)$ (hence, by Proposition 5.4, in $(p^*(\mathcal{M}^r))_s(Y)$).

Proof. Testing with $\phi \in Y'$ reduces the problem to showing that

$$\phi(g(s,t)) - \sum D^\gamma(\phi \circ g)(t) \cdot h_\gamma(s) \tag{10}$$

is in $p^*(\mathcal{M}^r)$, but this follows from Proposition 5.7, applied to $f = \phi \circ g$.

Theorem 5.1 *If J is a semi-Weil ideal, and Y a CVS, $J_s(Y) = J_w(Y)$ (as linear subspaces of $C^\infty(\mathbf{R}^{l+k}, Y)$).*

Proof. Let $g = g(s,t)$, $g : \mathbf{R}^l \times \mathbf{R}^k \to Y$, be a map in $J_w(Y)$. Since the $h_\gamma(s)$ are in I, the sum $\sum (D^\gamma g)(t) \cdot h_\gamma(s)$ in (9) is in $J_s(Y)$. The whole expression in (9) is in $(p^*(\mathcal{M}^r))_w(Y)$, by Proposition 5.8, and hence, by Proposition 5.4, in $(p^*(\mathcal{M}^r))_s(Y)$ which in turn is contained in $J_s(Y)$. This proves the Theorem.

From now on, we write $J(Y)$ instead of $J_w(Y)$ or $J_s(Y)$, in case J is a semi-Weil ideal and Y a CVS; for, they agree, by the Theorem.

We now discuss the description of semi-Weil ideals in terms of differential operators.

If $I \subseteq C^\infty(\mathbf{R}^n)$ is an ideal which is the null space of a family of differential operators $\{d^\beta \mid \beta \in B\}$ (not necessarily supported at the same $x \in \mathbf{R}^n$), then it follows from Proposition 5.1 that $I_w(Y) \subseteq C^\infty(\mathbf{R}^n, Y)$ is the null space of the family of the d^β_Y.

If I is a Weil ideal in $C^\infty(\mathbf{R}^l)$, null space of a finite family $\{d^\beta \mid \beta \in B\}$ of differential operators supported at $0 \in \mathbf{R}^l$, then $J \subseteq C^\infty(\mathbf{R}^{l+k})$ is the null space of the (infinite) family of differential operators $d^{\beta,x}$, $\beta \in B$, $x \in \mathbf{R}^k$, where for a function $f(s,t) \in C^\infty(\mathbf{R}^{l+k})$, $d^{\beta,x}(f)$ takes the relevant partial derivatives in the s-directions, and then substitutes 0 for s and x for t.

It follows that $J(Y)$, for Y a CVS, may be described as the null space of the $B \times \mathbf{R}^k$-indexed family of differential operators $d^{\beta,x}_Y : C^\infty(\mathbf{R}^{l+k}, Y) \to Y$.

Also, it follows that under the transposition isomorphism $C^\infty(\mathbf{R}^{l+k}, Y) \cong C^\infty(\mathbf{R}^l, C^\infty(\mathbf{R}^k, Y))$, the linear subspace $J(Y)$ on the left corresponds to the linear subspace $I(C^\infty(\mathbf{R}^k, Y)$ on the right.

Let $I \subseteq \mathbf{R}^l$ be a Weil ideal, $I \supseteq \mathcal{M}^r$. Let $\{D^\beta \mid \beta \in B\}$ be a family of differential operators at 0, of degree $< r$, forming a basis for $(C^\infty(\mathbf{R}^l)/I)^*$. Note that B is a finite set. Let the dual basis for $(C^\infty(\mathbf{R}^l)/I$ be represented by polynomials of degree $< r$, $\{p_\beta(s) \mid \beta \in B\}$. Then we can construct a linear isomorphism

$$C^\infty(\mathbf{R}^l, Y)/I(Y) \to \prod_B Y,$$

by sending the class of $f : \mathbf{R}^l \to Y$ into the B-tuple $D^\beta_Y(f)$. Its inverse is given by sending a B-tuple $y_\beta \in Y$ to $\sum_B p_\beta(s) \cdot y_\beta$.

It follows that for a semi-Weil ideal $J = p^*(I) \subseteq \mathbf{R}^{l+k}$, as above,

$$C^\infty(\mathbf{R}^{k+l}, Y)/J(Y) \cong \prod_B C^\infty(\mathbf{R}^k, Y). \tag{11}$$

(The isomorphism is not canonical but depends on the choice of a linear basis $p_\beta(s)$ for the Weil algebra $C^\infty(\mathbf{R}^l)/I$.)

6 Cahiers Topos

The site $\underline{D}$ of definition of this topos $\mathcal{C}$ is the dual of the category of C^∞-rings of the form $C^\infty(\mathbf{R}^{l+k})/J$ where J is a semi-Weil ideal, coming from a Weil ideal $I \subseteq C^\infty(\mathbf{R}^l)$. – There is a full embedding $i : \underline{Mf} \to \mathcal{C}$.

The full embedding h, described in [14], of $\underline{Con}^\infty$ into $\mathcal{C}$ is, on objects, given by sending a CVS X into the presheaf on $\underline{D}$ given by

$$C^\infty(\mathbf{R}^{l+k})/J \mapsto C^\infty(\mathbf{R}^{l+k}, X)/J(X).$$

For smooth maps $X \to Y$, composing with Y preserves the property of "being congruent mod J", cf. Prop. 2.1 in [12] or Coroll. 2 in [14], and this describes the functorality. For finite dimensional vector spaces X, $h(X) = i(X)$.

The embedding h is full. It preserves the exponentials in CVS, and furthermore, if X is a CVS, the R-module $h(X)$ in $\mathcal{C}$ "satisfies the vector form of Axiom 1" (generalized Kock-Lawvere Axiom), so that in particular synthetic calculus for curves $R \to h(X)$ is available; cf. the final remark in [12]. From this, one may deduce that the embedding h preserves differentiation, i.e. for $f : \mathbf{R} \to X$ a smooth curve, its derivative $f' : \mathbf{R} \to X$ goes by h to the synthetically defined derivative of the curve $h(f) : R = h(\mathbf{R}) \to h(X)$. This follows by repeating the argument for Theorem 1 in [9] (the Theorem there deals with the case where the codomain of f is $\mathbf{R}$, but it is valid for X as well because $h(X)$ satisfies the vector form of Axiom 1).

We note the following aspect of the embedding h. Let X be a CVS. Each $\phi \in X'$ is smooth linear $X \to \mathbf{R}$ and hence defines a map $h(\phi) : h(X) \to h(\mathbf{R}) = R$ in $\mathcal{C}$. This map is R-linear.

Proposition 6.1 *The maps $h(\phi) : h(X) \to R$, as ϕ ranges over X', form a jointly monic family.*

Proof. The assertion can also be formulated: the natural map

$$e : h(X) \to \prod_{\phi \in X'} R$$

is monic (where $proj_\phi \circ e := h(\phi)$). To prove that this (linear) map is monic, consider an element a of the domain, defined at stage $C^\infty(\mathbf{R}^{l+k})/J$, where J is a semi-Weil ideal. So $a \in C^\infty(\mathbf{R}^{l+k}, X)/J(X)$. Let $\alpha \in C^\infty(\mathbf{R}^{l+k}, X)$ be a smooth map representing the class a, $a = \alpha + J(X)$. The element $e(a)$ is the X' tuple $a_\phi + J(X)$, where $a_\phi \in C^\infty(\mathbf{R}^{l+k})/J(X)$ is represented by the

smooth map $\phi \circ \alpha : \mathbf{R}^{l+k} \to \mathbf{R}$. To say a maps to 0 by e is thus to say that for each $\phi \in X'$, $\phi \circ \alpha \in J$. But this is precisely the defining property for α itself to be in $J_w(X) = J(X)$, i.e. for a to be the zero as an element of $h(X)$ (at the given stage $C^\infty(\mathbf{R}^{l+k})/J$).

7 The internal space of test functions

We first analyze the object R^R in $\mathcal{C}$. Because h preserves exponentials, and $R = i(\mathbf{R}) = h(\mathbf{R})$, R^R is $h(C^\infty(\mathbf{R}))$. Therefore an element of R^R at stage $C^\infty(\mathbf{R}^{l+k})/J$, where $J = p^*(I)$ is a semi-Weil ideal as above, is an element of

$$C^\infty(\mathbf{R}^{l+k}, C^\infty(\mathbf{R}))/J(C^\infty(\mathbf{R})) \cong \prod_B C^\infty(\mathbf{R}^k, C^\infty(\mathbf{R})) \cong \prod_B C^\infty(\mathbf{R}^{k+1}),$$

by (11).

In concrete terms, an element of R^R, defined at stage $C^\infty(\mathbf{R}^{l+k})/J$, is thus given by the class mod J^* of a smooth map $f : \mathbf{R}^{l+k+1} \to \mathbf{R}$, $f(s,t,x)$, and even more concretely, by the B-tuple of smooth maps $\mathbf{R}^{k+1} \to \mathbf{R}$, $D^\beta f(0,t,x)$ (recall that the D^β's differentiate in the s-variable only, and then substitute $s = 0$).

The following is a formula with a free variable f that ranges over R^R:

$$\exists b > 0[\ \ \forall x, (x < -b \vee x > b) \Rightarrow f(x) = 0\ \ ]. \tag{12}$$

Let us write $|x| > b$ as shorthand for the formula $x < -b \vee x > b$ (so, in spite of the notation, we don't assume an "absolute value" function). Then the formula (12) gets the more readable appearance:

$$\exists b > 0[\ \ \forall x, |x| > b \Rightarrow f(x) = 0\ \ ]. \tag{13}$$

(verbally: "f is a function $R \to R$ of bounded support" (namely support contained in the interval $[-b, b]$). Its extension is a subobject $\mathcal{D}(R) \subseteq R^R$.

We shall as a preliminary investigate when an element of R^R defined at a stage of the particular form $C^\infty(\mathbf{R}^k)$ belongs to the internal $\mathcal{D}(R)$ (described as the extension of the formula (13)). So for the present, there are no Weil ideals invloved.

For simplicity of notation, let us write K for $i(\mathbf{R}^k) = R^k$.

So consider an element $f \in_K R^R$. This means a map $K \to R^R$ in $\mathcal{C}$, and this in turn corresponds, by transposition, and by fullness of the embedding i, to a smooth map

$$\hat{f} : \mathbf{R}^k \times \mathbf{R} \to \mathbf{R}.$$

Now we have that

$$\vdash_K \exists b > 0[\forall x, |x| > b \Rightarrow f(x) = 0]$$

if and only if there is a covering U_i of K ($i \in I$) and witnesses $b_i \in_{U_i} R_{>0}$, so that for each i

$$\vdash_{U_i} \forall x,\ |x| > b_i \Rightarrow f(x) = 0$$

Externally, this implies that $b_i : U_i \to \mathbf{R}$ is a smooth function with positive values, with the property that for all $t \in U_i$, if x has $x > b_i(t)$, then $\hat{f}(t, x) = 0$. The following Lemma then implies that f is of l.u.b.s. on U_i, and since the U_i's cover K, f is of l.u.b.s. on K.

Lemma 7.1 *Let $g : U \times \mathbf{R} \to \mathbf{R}$ have the property that there exists a smooth (or just continuous) $b : U \to R_{>0}$ so that for all $t \in U$ $|x| > b(t)$ implies $g(t, x) = 0$. Then g is l.u.b.s.*

Proof. For each $t \in U$, let c_t denote $b(t) + 1$. There is a neighbourhood V_t around t such that $b(y) < c_t$ for all $y \in V_t$. The family of V_t's, together with the constants c_t now witness that g is l.u.b.s. For, for all $y \in V_t$ and any x with $|x| > c_t$, we have $|x| > c_t > b(y)$, so $g(y, x) = 0$.

Conversely, if $\hat{f}$ is l.u.b.s., it is easy to see that the element $f \in_K R^R$ satisfies the formula (reduce to the uniformly bounded case, and write the condition as existence of a commutative square).

So we conclude that for $f \in_K R^R$, $f \in_K \mathcal{D}(R)$ iff the external function $f : K \times \mathbf{R} \to \mathbf{R}$ is l.u.b.s., i.e., by Theorem 2.2, iff $\hat{f} : K \to C^\infty(\mathbf{R})$ factors by a (diffeologically!) smooth map through the inclusion $\mathcal{D}(\mathbf{R}) \subseteq C^\infty(\mathbf{R})$, i.e. belongs to $C^\infty(\mathbf{R}^k, \mathcal{D}(\mathbf{R})) = h(\mathcal{D}(\mathbf{R}))(C^\infty(\mathbf{R}^k))$. This proves that, at least as far as generalized elements, defined at stages where no Weil ideal is involved, we have "$h(\mathcal{D}(\mathbf{R})) = \mathcal{D}(R)$", more precisely,

$$h(\mathcal{D}(\mathbf{R}))(C^\infty(\mathbf{R}^k)) = \mathcal{D}(R)(C^\infty(\mathbf{R}^k)). \tag{14}$$

To get a similar conclusion for elements of $\mathcal{D}(R)$ (as synthetically defined by (13)), defined at stage $C^\infty(\mathbf{R}^{l+k})/J$, we shall prove that such can be represented by B-tuples of elements defined at stage $C^\infty(\mathbf{R}^k)$; we shall prove that such a B-tuple defines an element of $\mathcal{D}(R)$ precisely if each of these B elements is an element in $\mathcal{D}(R)$. This proof is a piece of "purely synthetic reasoning":

We consider an $\mathbf{R}$-algebra object R in a topos $\mathcal{C}$, and assume that R satisfies the general "Kock-Lawvere" (K-L) axiom (recalled below), and is equipped with a strict order relation $<$. Because the reasoning is purely synthetic, we don't have to think in terms of sheaf semantics, so for instance we don't have to be specific at what "stages", the "elements" in question are defined; we reason *as if* all elements are global elements. For $b > 0$, we write $|x| > b$ as shorthand for $x < -b \vee x > b$ as before; and we stress again that we don't assume any absolute-value function (it does not exist in the Cahiers topos). We argue in $\mathcal{C}$ as if it were the category of sets, making sure to use only intuitionistically valid reasoning.

A Weil algebra $C^\infty(\mathbf{R}^l)/I$, as above, gives rise to an "infinitesimal" subobject $W \subseteq R^l$: pick a (finite) set of differential operators D_β ($\beta \in B$) forming a basis for $(C^\infty(\mathbf{R}^l)/I)^*$, and take the dual basis for $C^\infty(\mathbf{R}^l)/I$, whose elements are represented mod I by polynomials $p_\beta(s)$ in l variables. Then $W \subseteq R^l$ is the extension of the formulas $p_\beta(s) = 0$, s being a variable ranging over R^l (note that real polynomials in l variables define functions $R^l \to R$ in $\mathcal{C}$).

We assume that such W's are internal atoms, in a sense we partially recall below; this is so for all interesting models $\mathcal{C}$ of SDG, including the Cahiers Topos.

To say that an R-module object Y in $\mathcal{C}$ satisfies the general K-L axiom is to say that for each such Weil algebra, the map

$$\prod_B Y \to Y^W$$

given by

$$(y_\beta)_{\beta \in B} \mapsto [s \mapsto \sum_B p_\beta(s) \cdot y_\beta]$$

is an isomorphism.

We assume that R itself satisfies K-L. This immediately implies that R^M does for any $M \in \mathcal{C}$. We shall consider R^R.

Now recall that $\mathcal{D}(R) \subseteq R^R$ was the subobject which is the extension of the formula (13) (with free variable f ranging over R^R) $\exists b > 0 : |x| > b \Rightarrow f(x) = 0$.

Proposition 7.1 *Let a B-tuple of elements f_β in R^R represent an element in $(R^R)^W$. Then it defines an element in the sub"set" $(\mathcal{D}(R))^W$ if and only if each f_β is in $\mathcal{D}(R)$.*

Proof. Assume first that all f_β are in $\mathcal{D}(R)$. For each β there exists a witnessing $b_\beta > 0$ witnessing that the formula (13) holds for f_β, but since there are only finitely many β's, we may assume one common witness $b > 0$. So for all β, and for all x with $|x| > b$, $f_\beta(x) = 0$. But then for each such x, the function of $s \in W$ given by

$$s \mapsto \sum_\beta p_\beta(s) \cdot f_\beta(x)$$

is the zero function. The sum here, as a function of s and x, is the element of $(R^R)^W$ corresponding to the B-tuple f_β, and for $|x| > b$, it is the zero. So for each s, the given fixed b witnesses that the sum, as a function of x, is in $\mathcal{D}(R)$.

Conversely, assume that the f_β's are such that the corresponding function $W \to R^R$ factors through $\mathcal{D}(R)$. So for each $s \in W$, the function

$$x \mapsto \sum_\beta p_\beta(s) \cdot f_\beta(x)$$

belongs to $\mathcal{D}(R)$. So

$$\forall s \in W \; \exists b > 0 : |x| > b \Rightarrow \sum_\beta p_\beta(s) \cdot f_\beta(x) = 0. \tag{15}$$

We would like to *pick* for each $s \in W$ a $\tilde{b}(s)$ such that

$$\forall s \in W : |x| > \tilde{b}(s) \Rightarrow \sum_\beta p_\beta(s) \cdot f_\beta(x) = 0;$$

the existence of such a *function* $\tilde{b}$ follows from (15) by a use of the Axiom of Choice, so in general is not possible in a topos. But since W is an internal atom, and s ranges over W, such a function $\tilde{b}$ exists after all. (See the Appendix for a general formulation and proof of this principle.)

But now $|x| > \tilde{b}(0) \Rightarrow |x| > \tilde{b}(s)$ for all $s \in W$, because $\tilde{b}$, as does any function, preserves infinitesimals, and because strict inequality is unaffected by infinitesimals. So we have a b, namely $\tilde{b}(0)$, so that

$$\forall s \in W : |x| > b \Rightarrow \sum p_\beta(s) \cdot f_\beta(x) = 0.$$

So for $|x| > b$,

$$\forall s \in W, \sum p_\beta(s) \cdot f_\beta(x) = 0.$$

Thus, for fixed x with $|x| > b$, the function of s here is constantly 0. But functions $W \to R$ can uniquely be described as linear combinations of the $p_\beta(s)$'s (this is a verbal rendering of the K-L axiom for R). So for such x each $f_\beta(x)$ is 0. So b witnesses, for each β, that $f_\beta \in \mathcal{D}(R)$. This proves the Proposition.

Combining (11) (with $\mathcal{D}(\mathbf{R})$ for Y) with (14) and Proposition 7.1, we get

Theorem 7.1 *The subobject $\mathcal{D}(R)$ of R^R is exactly $h(\mathcal{D}(\mathbf{R}))$.*

The R-module $\mathcal{D}(R) \subseteq R^R$ (=the extension of the formula (13)) is by definition the internal vector space of test functions; and we form the subobject

$$\mathcal{D}'(R) \subseteq R^{\mathcal{D}(R)}$$

which is the extension of the formula "ϕ is R-linear" (ϕ a variable ranging over $R^{(\mathcal{D}(R)}$). So $\mathcal{D}'(R)$ is the internal vector space of distributions.

We make an analysis of $h(Y')$ for a general CVS Y. Recall that the diffeology on Y' is inherited from that of $C^\infty(Y, \mathbf{R})$, so that (for an open $U \subseteq \mathbf{R}^k$), the smooth plots $U \to Y'$ are in bijective correspondence with smooth maps $U \times Y \to \mathbf{R}$, which are $\mathbf{R}$-linear in the second variable $y \in Y$. It follows that the elements at stage $C^\infty(\mathbf{R}^k)$ (no Weil ideal involved) are in bijective correspondence with smooth maps $\mathbf{R}^k \times Y \to \mathbf{R}$, $\mathbf{R}$-linear in the second variable, or equivalently, with smooth $\mathbf{R}$-linear maps $Y \to C^\infty(\mathbf{R}^k, \mathbf{R})$, i.e. with smooth $\mathbf{R}$-linear maps $Y \to C^\infty(\mathbf{R}^k)$.

On the other hand, an element of $R^{h(Y)}$ defined at stage $C^\infty(\mathbf{R}^k)$ is a morphism $R^k \to R^{h(Y)}$, hence by double transposition it corresponds to a map $h(Y) \to R^{R^k}$; and it belongs to the subobject $Lin_R(h(Y), R)$ iff its double transpose is R-linear. Since h is full and faithful, and preserves the cartesian closed structure (hence the transpositions), this double transpose corresponds bijectively to a smooth map $Y \to C^\infty(\mathbf{R}^k, \mathbf{R}) = C^\infty(\mathbf{R}^k)$, and R-linearity is equivalent to $\mathbf{R}$-linearity, by the following general

Lemma 7.2 *Let X and Y be CVS's. Then a smooth map $f : Y \to X$ is $\mathbf{R}$-linear iff $h(f) : h(Y) \to h(X)$ is R-linear.*

Proof. The implication $\Rightarrow$ is a consequence of the fact that h preserves binary cartesian products (and of $h(\mathbf{R}) = R$). For the implication $\Leftarrow$, we just apply the global sections functor Γ; note that $\Gamma(Y)$ is the underlying set of the vector space Y, and similar for X; and $\Gamma(R) = \mathbf{R}$.

We have in particular:

Proposition 7.2 *There is a natural one-to one correspondence between distributions on* $\mathbf{R}$*, and* R*-linear maps* $\mathcal{D}(R) \to R$

Proof. By fullness of the embedding h of $\underline{Con}^\infty$ into the Cahiers topos $\mathcal{C}$, there is a bijection between the set of smooth maps $\mathcal{D}(\mathbf{R}) \to \mathbf{R}$, and the set of morphisms in $\mathcal{C}$, $h(\mathcal{D}(\mathbf{R})) \to h(\mathbf{R}) = R$, and $\mathbf{R}$-linearity corresponds to R-linearity, by the above Lemma. The result now follows from $h(\mathcal{D}(\mathbf{R})) = \mathcal{D}(R)$ (Theorem 7.1).

This result should be compared to the Theorem of [21], or Proposition II.3.6 in [18], where a related assertion is made for distributions-with-compact-support, i.e. where $\mathcal{D}(R)$ is replaced by the whole of R^R, – or even with R^M, with M an arbitrary smooth manifold. Distributions with compact support are generally easier to deal with synthetically (as we did in [16]), but they are not adequate for the heat equation.

8 Half Line

By Theorem 1.1, the two C^∞-rings $C^\infty(\mathbf{R})/\mathcal{M}^\infty_{\geq 0}$ and $C^\infty(\mathbf{R}_{\geq 0})$ are isomorphic, where $\mathcal{M}^\infty_{\geq 0}$ is the ideal of smooth functions vanishing on the non-negative half line, and $C^\infty(\mathbf{R}_{\geq 0})$ is the ring of smooth functions $\mathbf{R}_{\geq 0} \to \mathbf{R}$. Being a quotient of the ring $C^\infty(\mathbf{R})$ which represents $R \in \mathcal{C}$, it defines a subobject of R, which we denote $R_{\geq 0}$ (also considered in [13][2]). – Thus, $R_{\geq 0}$ is "represented from the outside" by the C^∞-ring $C^\infty(\mathbf{R})/\mathcal{M}^\infty_{\geq 0} \cong C^\infty(\mathbf{R}_{\geq 0})$.

Proposition 8.1 *Let* $I \subseteq C^\infty(\mathbf{R}^l)$ *be a Weil ideal and let* $f : \mathbf{R}^l \times \mathbf{R}^n \to \mathbf{R}$ *be a smooth function. Then the following are equivalent:*

1. $f(\underline{0}, x) \geq 0$ *for all* $x \in \mathbf{R}^n$
2. $\rho(f(\underline{w}, x)) \in I^*$ *for all* $\rho \in m^\infty_{\mathbf{R}_{\geq 0}}$.

[2]The ring representing $R_{\geq 0}$, was in loc.cit. defined using the ideal $\mathcal{M}^g_{\geq 0}$ of functions vanishing on an open neighbourhood of $\mathbf{R}_{\geq 0}$, rather than the ideal $\mathcal{M}^\infty_{\geq 0}$ considered here. But it can be proved that they represent (from the outside) the same object in the Cahiers topos.

Proof: "not 1" implies "not 2"; for, if $f(0,x) < 0$, we may find a function ρ vanishing on $\mathbf{R}_{\geq 0}$ and with value 1 at $f(0,x)$. Then $f \notin I^*$ (recall that any Weil ideal I consists of functions vanishing at 0).

1 implies 2: By Taylor expansion,

$$\begin{array}{rcl}(\rho \circ f)(\underline{w}, x) & = & (\rho \circ f)(\underline{0}, x) + \sum_i w_i (\rho \circ f)'_i(\underline{0}.x) + \\ & & \sum_{i,j} w_i w_j (\rho \circ f)'_{i,j}(\underline{0}, x) + \ldots\end{array}$$

where $(-)_i = \partial/\partial x_i,\ (-)_{i,j} = \partial^2/\partial x_i x_j$ etc.

This series finishes after finitely many terms modulo I^*, since a product of powers of w_i's belong to the ideal I. But each of its terms is 0: Indeed, so is the term without derivatives, by hypothesis. But so are the others. For instance. $(\rho \circ f)'_i(\underline{0}, x) = \rho'(f(\underline{0}, x)\partial f/\partial x_i(\underline{0}, x)$ is 0, since the derivative of ρ is zero on non-negative reals (by definition of $m^{\infty}_{\mathbf{R}_{\geq 0}}$).

An element F of $R_{\geq 0}$ defined at stage $C^{\infty}(\mathbf{R}^{l+k})/J$ is represented by a function f satisfying the conditions of the Proposition.

Proposition 8.2 *There is a bijection between the set of maps $K : \mathbf{R}_{\geq 0} \to X$ which are strongly smooth (in the sense of Definition 1.1 in Section 1), and the set of maps $\overline{K} : R_{\geq 0} \to h(X)$ in $\mathcal{C}$.*

Proof/Construction. Given an element F of $R_{\geq 0}$ defined at stage $C^{\infty}(\mathbf{R}^{l+k})/J$, represented by f, as above. We want to produce an element $K(F)$ of X defined at stage $C^{\infty}(\mathbf{R}^{l+k})/J$, in other words, an element of $C^{\infty}(\mathbf{R}^{l+k}, X)/J(X)$. We take the element represented (mod $J(X)$) by the smooth map $\mathbf{R}^{l+k} \to X$ given by the r first terms of the series $(s,t) \mapsto$

$$K(f(0,t)) + K'(f(0,t)) \cdot [f(s,t) - f(0,t)] + \frac{K''(f(0,t))}{2!} \cdot [f(s,t) - f(0,t)]^2 + \ldots.$$

Note that $[f(s,t) - f(0,t)]^r \in J$ since for fixed t, $f(s,t) - f(0,t) \in \mathcal{M}$, hence $[f(s,t) - f(0,t)]^r \in \mathcal{M}^r \subseteq I$. This suffices, by the description of semi-Weil ideals in terms of differential operators.

We have to prove that the class mod $J(X)$ of this map only depends on the class of f. If we change f into $f + h$ with $h \in J$, the term in the square brackets (real numbers!) change into $[f(s,t) - f(0,t) + h(s,t)]$ (using that $h(0,t) = 0$), and the "Taylor coefficients" ($\in X$) do not change at all, for the same reason. – Uniqueness is easy, using Proposition 6.1, together with the fullness result from [20] on manifolds with boundary.

The Proposition is a "mixed fullness" result; we have that Con^∞ and $\underline{Mf}$ (= smooth manifolds), (even the category of smooth manifolds with boundary), embed fully in The Cahiers Topos; but at present we do not have a general result about what can be said about $C^\infty(M, X)$, for M a manifold (possibly with boundary) and X a CVS – not to speak of $C^\infty(X, M)$.

9 Heat Equation in the Cahiers Topos

For any topos $\mathcal{C}$ with a ring object R with a preorder $\leq$, we may form the R-module $\mathcal{D}'(R^n)$ of distributions on R^n, as explained in Section 7. If $\mathcal{C}, R$ is a model of SDG, then $\mathcal{D}'(R^n)$ automatically satisfies the "vector form" of the general Kock-Lawvere axiom, so that (synthetic) differentiation of functions $K : R \to \mathcal{D}'(R^n)$ is possible - it is even enough that K be defined on suitable ("formally etale") subobjects of R, like $R_{\geq 0}$. We think of the domain R or $R_{\geq 0}$ as "time", and denote the differentiation of curves K w.r. to time by the Newton dot, $\dot{K}$. On the other hand, we think of R^n as a space, and the various partial derivatives $\partial/\partial x_i$ $(i = 1, \ldots, n)$, as well as their iterates, we call spatial derivatives; in case $n = 1$, they are just denoted $(-)'$, $(-)''$, etc. They live on $\mathcal{D}'(R^n)$ as well, by the standard way of differentiating distributions (which immediately translates into the synthetic context, cf. e.g. [16]). The heat equation for (Euclidean) space in n dimensions says $\dot{K} = \Delta \circ K$, where Δ is the Laplace operator; in one dimension it is thus the equation

$$\dot{K} = K''.$$

We can summarize the constructions into an general existence theorem about models for SDG:

Theorem 9.1 *There exists a well-adapted model for SDG (with a preorder $\leq$ on R), in which the heat equation on the (unlimited) line R has a unique solution $k : R_{\geq 0} \to \mathcal{D}'(R)$ with initial value $k(0) = \delta(0)$ (the Dirac distribution).*

Proof. The well adapted model witnessing the validity of the Theorem is the Cahiers Topos $\mathcal{C}$. Consider the classical heat kernel, viewed, as we did in Section 4, as a map $\mathbf{R}_{\geq 0} \to \mathcal{D}'(\mathbf{R})$. By Theorem 4.1, this map is smooth in the strong sense, hence by Proposition 8.2, it defines a morphism in $\mathcal{C}$, $\bar{K} : R_{\geq 0} \to h(\mathcal{D}'(\mathbf{R})$. This $\bar{K}$ is going to be our k. By Theorem (7.1)), its

codomain is the desired $\mathcal{D}'(R)$. So all that remains is to prove that this k satisfies the heat equation $\dot{k} = \Delta \circ k$. This is a purely formal argument from the fact that K does, and the fact that h takes "analytic" differentiation into the "synthetic" differentiation in $\mathcal{C}$. We give this argument. Synthetically, we want to prove that for all $x \in \mathbf{R}_{\geq 0}$ and $d \in D$

$$k(x+d) = k(x) + d \cdot \Delta(k(x)).$$

Universal validity of this equation means that a certain diagram, with domain $R_{\geq 0} \times D$ and codomain $\mathcal{D}'(R)$, commutes. Taking the transpose of this diagram, we get a diagram with domain $R_{\geq 0}$ and codomain $(\mathcal{D}'(R))^D \cong \mathcal{D}'(R) \times \mathcal{D}'(R)$ (by K-L for $\mathcal{D}'(R)$):

$$\begin{tikzcd}
R_{\geq 0} \arrow[r, "\hat{+}"] \arrow[d, "k"'] & (R_{\geq 0})^D \arrow[dd, "k^D"] \\
\mathcal{D}'(R) \arrow[d, "(1,\Delta)"'] & \\
\mathcal{D}'(R) \times \mathcal{D}'(R) & (\mathcal{D}'(R))^D \arrow[l, "\cong"']
\end{tikzcd}$$

When the global sections functor Γ is applied to this diagram, the left hand column yields $(K, \Delta \circ K)$, because $\Gamma(k) = K$; the composite of the other maps is $(K, \dot{K})$ because Γ takes synthetic differentiation into usual differentiation. Since K satisfies $\dot{K} = \Delta \circ K$, we conclude that Γ applied to the exhibited diagram commutes. Now Γ is not faithful, but because of the special form of the domain and codomain of the two maps to be compared, we may still get the conclusion, by virtue of the following

Proposition 9.1 *Given a map $a : R_{\geq 0} \to h(X)$, where X is a CVS. If $\Gamma(a) = 0$, then $a = 0$.*

Proof. Since the $h(\phi) : h(X) \to R$ are jointly monic as ϕ ranges over X', by Proposition 6.1, it suffices to see that each $h(\phi) \circ a$ is 0. Since $\Gamma(h(\phi) \circ$

$a) = \phi \circ \Gamma(a)$, this reduces the question to the case where $X = \mathbf{R}$. A map $a : R_{\geq 0} \to R$ is tantamount to an element in $\bar{a} : C^\infty(\mathbf{R}_{\geq 0})$, and the assumption $\Gamma(a) = 0$ is tantamount to $\bar{a}(t) = 0$ for all $t \in \mathbf{R}_{\geq 0}$. But this clearly implies that $\bar{a}$, and hence a, is 0.

Appendix

Recall that an *atom* A in a cartesian closed category $\mathcal{C}$ is an object so that the exponential functor $(-)A$ has a right adjoint; in particular, it takes epimorphisms to epimorphisms. The following says that "axiom of choice" holds for "A"-tuples sets:

Proposition 9.2 *Assume that A is an atom, B arbitrary and $R \subseteq A \times B$. Then*

$$(\forall a \in A)(\exists b \in B)\ R(a,b) \Longrightarrow (\exists \tilde{b} \in B^A)(\forall a \in A)\ R(a, \tilde{b}(a))$$

Proof: The hypothesis means that the composite $R \to A \times B \xrightarrow{\pi_1} A$ is surjective. By exponentiation, and the assumption that A is an atom, the composite $R^A \to A^A \times B^A \xrightarrow{\pi_1} A^A$ is surjective. In particular, $1_A \in A^A$ must have a pre-image $(1_A, \tilde{b})$. This $\tilde{b}$ obviously does the job.

References

[1] Artin, M., Grothendieck, A. and Verdier, J.-L., Théorie des Topos (SGA 4), Springer LNM 269 (1972).

[2] Chen, K.T. Iterated integrals of differential forms and loop space homology, Annals of Math. (2) 97 (1972), 217-146.

[3] Dubuc, E., Sur les modèles de la géométrie différentielle synthétique, Cahiers de Topologie et Géometrie Diff. 20 (1979), 231-279.

[4] Frölicher, A., Cartesian Closed Categories and Analysis of Smooth Maps, in "Categories in Continuum Physics", Buffallo 1982 (ed. F.W. Lawvere and S.H. Schanuel), Springer Lecture Notes in Math. 1174 (1986).

[5] Frölicher, A., Gisin, B., Kriegl, A., General differentiation theory, in Aarhus Var. Publ. Series No. 35 (1983), 0125-153.

[6] Frölicher, A., Kriegl, A., Linear Spaces and Differentiation Theory, Wiley 1988.

[7] Horváth, J., Topological Vector Spaces and Distributions, Vol. 1, Addison-Wesley 1966.

[8] Kelley, J.L., General Topology, Van Nostrand 1955.

[9] Kock, A. Properties of well-adapted models for synthetic differential geometry, Journ. Pure Appl. Alg. 20 (1981), 55-70.

[10] Kock, A., Synthetic Differential Geometry, Cambridge University Press 1981.

[11] Kock, A., Calculus of smooth functions between convenient vector spaces, Aarhus Preprint Series 1984/85 No. 18. Retyped in http://home.imf.au.dk/kock/CSF.pdf

[12] Kock, A., Convenient vector spaces embed into the Cahiers topos, Cahiers de Topologie et Géométrie Diff. Catégoriques 27 (1986), 3-17.

[13] Kock, A. and Reyes, G.E., Models for synthetic integration theory, Math. Scand. 48 (1981), 145-152.

[14] Kock, A. and Reyes, G.E., Corrigendum and addenda to "Convenient vector spaces embed", Cahiers de Topologie et Géométrie Diff. Catégoriques 28 (1987), 99-110.

[15] Kock, A. and Reyes, G.E., Some differential equations in SDG, arXiv:math.CT/0104164 .

[16] Kock, A. and Reyes, G.E., Some calculus with extensive quantities: wave equation, Theory and Applications of Categories, Vol. 11 (2003), No. 14.

[17] Losik, M.V. Fréchet manifolds as diffeological spaces, Soviet Math. (Iz. vuz) 5 (1992), 36-42.

[18] Moerdijk, I. and and Reyes, G.E., Models for Smooth Infinitesimal Analysis, Springer 1991.

[19] Nel, L.D. Enriched locally convex structures, Differential Calculus and Riesz representations, Journ. Pure Appl. Alg. 42 (1986), 165-184.

[20] Porta, H. and Reyes, G.E., Variétés à bord et topos lisse, in "Analyse dans les topos lisses" (ed. G.E. Reyes), Rapport de Rechercehes D.M.S. 80-12, Université de Montréal 1980.

[21] Quê, N.V. and Reyes, G.E. Théorie des distribution et théorème d'extension de Whitney, in "Analyse dans les topos lisses" (ed. G.E. Reyes), Rapport de Rechercehes D.M.S. 80-12, Université de Montréal 1980.

[22] Schwartz, L., Méthodes mathématiques pour les sciences physiques, Hermann Paris 1961.

[23] Schwartz, L., Théorie des distributions, Tome 1, Hermann Paris 1957.

[24] Seeley, R.T. , Extension of C^∞ functions defined in a half space, Proc. Amer. Math. Soc. 15 (1964), 625-626.

[25] Souriau, J.-M. Groupes différentielles et physique mathémathique, feuilletages et quantification géométriques, Coll. Travaux en Cours, Hermann Paris 1984, 75-79.

[26] Torre, C.A., Differentiability, Convenient Spaces and Smooth Diffeologies, arXiv:math.DG/0112036 .