Lower Bounds for
External Memory Dictionaries

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Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms
Baltimore, MD, USA, January 13, 2003
Dictionary

- **Queries**
  - membership
  - predecessor / successor
  - range queries ...

- **Updates**
  - insertions
  - deletions
Dictionary

- **Queries**
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**This talk**: Comparison based, membership, insertions
# Dictionaries – Comparison Based

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Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
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Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
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<td>Borodin et al. 1982</td>
<td>$O(t)$</td>
<td>$\Rightarrow N/2^{O(t)}$</td>
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![Graph showing Insert and Search times for different models](https://via.placeholder.com/150)

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Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
External Memory Model

Aggarwal and Vitter 1988

- One I/O moves $B$ consecutive records from/to disk
- Cost: number of I/Os
- Elements can be copied and compared in internal memory

$N =$ problem size
$M =$ memory size
$B =$ I/O block size

Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
B-trees
– An External Memory Dictionary

Bayer and McCreight 1972

$O(B)$
B-trees
– An External Memory Dictionary

Bayer and McCreight 1972

\[ O \left( \log_B M \right) \]

\[ O \left( \log_B \frac{N}{M} \right) \]

\[ O(B) \]
B-trees
– An External Memory Dictionary

Bayer and McCreight 1972

\[ O \left( \log_B M \right) \]

\[ O \left( \log_B \frac{N}{M} \right) \]

\[ O(B) \]

Search/update path

Insert
Membership

\[ O \left( \log_B \frac{N}{M} \right) \] I/Os
# Dictionaries – External Memory

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Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
## Dictionaries – External Memory

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![Graph showing insert and search times for B-trees and Adversary](image-url)

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![Diagram](image_url)

Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
Comparisons vs. I/Os

Search vs. Insert

\[ \log N \quad \log \frac{N}{M} \]

\[ \log N \quad \text{Insert} \]

Aggarwal and Vitter 1988

Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
Comparisons vs. I/Os

Comparisons

\[ \Theta(N \log N) \]

I/Os

\[ \Theta\left(\frac{N}{B} \log_{M/B} \frac{N}{M}\right) \]

Aggarwal and Vitter 1988

Sorting

\[ \log B \frac{N}{M} \]

Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
Comparisons vs. I/Os

Comparisons

\[ \Theta(N \log N) \]

I/Os

\[ \Theta\left(\frac{N}{B} \log_{M/B} \frac{N}{M}\right) \]

Aggarwal and Vitter 1988

Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
Results

\[ \frac{N}{M \cdot \left( \frac{M}{B} \right)^{\Theta(\delta)}} \]

\[ \Theta(\log_\delta \frac{N}{M}) \]

\( \delta = \text{number of I/Os for } B \text{ insertions} \)

Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
Results

$N/(M \cdot (\frac{M}{B})^{\Theta(\delta)})$

$\Theta(\log_\delta \frac{N}{M})$

$\frac{1}{\varepsilon} \log_B \frac{N}{M}$

$\frac{B^\varepsilon}{\varepsilon} \log_B \frac{N}{M}$

$\delta = \text{number of I/Os for } B \text{ insertions}$

Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
Results

\[ \frac{N}{(M \cdot (\frac{M}{B})^{\Theta(\delta)})} \]

\[ \Theta(\log_{\delta} \frac{N}{M}) \]

\[ \frac{1}{\epsilon} \log \frac{N}{M} \]

\[ \log \frac{N}{M} \]

\[ \Theta(\log \frac{N}{M/B}) \]

\[ \log^{1+\epsilon} N \]

\[ \frac{B}{\epsilon} \log \frac{N}{M} \]

\[ \frac{B}{\log^3 N} \]

\[ B \log_B \frac{N}{M} \]

\[ \delta = \text{number of I/Os for } B \text{ insertions} \]

Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
Buffered B-trees
– how to speedup B-tree updates by a factor $B^{1-\varepsilon}$

Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
Buffered B-trees
– how to speedup B-tree updates by a factor $B^{1-\varepsilon}$

$O\left(\frac{1}{\varepsilon} \log_B M\right)$

$O\left(\frac{1}{\varepsilon} \log_B \frac{N}{M}\right)$

- B-tree with degree $\Theta(B^\varepsilon)$

Searches
$O\left(\frac{1}{\varepsilon} \log_B \frac{N}{M}\right)$

$B$ insertions
$O\left(\frac{B^\varepsilon}{\varepsilon} \log_B \frac{N}{M}\right)$
Buffered B-trees
– how to speedup B-tree updates by a factor $B^{1-\varepsilon}$

- B-tree with degree $\Theta(B^\varepsilon)$
- Buffers of $O(B)$ delayed insertions

\[
O\left(\frac{1}{\varepsilon} \log_B M\right) \quad \text{Internal Memory}
\]

\[
O\left(\frac{1}{\varepsilon} \log_B \frac{N}{M}\right) \quad \text{External Memory}
\]

Searches
- $O\left(\frac{1}{\varepsilon} \log_B \frac{N}{M}\right)$
- $B$ insertions
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Buffered B-trees

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Buffered B-trees
– how to speedup B-tree updates by a factor $B^{1-\varepsilon}$

- B-tree with degree $\Theta(B^\varepsilon)$
- Buffers of $O(B)$ delayed insertions
- On buffer overflow move $O(B^{1-\varepsilon})$ elements to a child with one I/O

\[
O \left( \frac{1}{\varepsilon} \log_B M \right) \}
\]

\[
O \left( \frac{1}{\varepsilon} \log_B \frac{N}{M} \right) \}
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Searches
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O \left( \frac{B^\varepsilon}{\varepsilon} \log_B \frac{N}{M} \right)
\]

Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
Lower Bound
– optimality of buffered B-trees

- Adversary online constructs $S_1, \ldots, S_K$
- Constructs $i$ such that $x_{ij}$ has not been in internal memory since $S_j$ was inserted, for all $j = 1, \ldots, K$
- $x_{i1}, \ldots, x_{iK}$ form an antichain, i.e. search requires $\geq K$ I/Os
Lower Bound (Cont.)

Let \( \hat{I} \) be the indexes \( i \) where

- \( x_{ij} \in S_j \) but is not in internal memory after inserting \( S_j \)
- \( x_{i1}, \ldots, x_{i(j-1)} \) have not been read into internal memory by the \( \delta \frac{|S_j|}{B} \) I/Os during the insertion of \( S_j \)

Construct \( I \subset \hat{I} \) such that all blocks in external memory contain \( O\left(\frac{B}{\delta}\right) \) elements \( x_{ij} \) where \( i \in I \)

- Existence by randomized sampling with probability \( O\left(1/\delta\right) \) and Chernoff bounds, provided \( B/\delta = \Omega(\log N) \)

Let \( x_{i(j+1)} \in S_j \) iff \( i \in I \)

\[ K = \Theta\left(\log_\delta \frac{N}{M}\right) \]
## Lower Bound — Below Sorting

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W.l.o.g. memory and each block totally ordered after each I/O

$N \log M + N B \log M B = N M |

Antichain of size $\binom{B}{2}$

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- W.l.o.g. memory and each block totally ordered after each I/O

\[ N \log M + \delta \frac{N}{B} \left( B \log \frac{M}{B} \right) \] comparisons

- Insert in internal memory
- Merging a block with internal memory
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- W.l.o.g. memory and each block totally ordered after each I/O

$$N \log M + \delta \frac{N}{B} \left( B \log \frac{M}{B} \right)$$ comparisons

- Insert in internal memory

- Merging a block with internal memory

- Antichain of size (Borodin et al. 1982 / Dillworth’s lemma)

$$\frac{N}{2^{\log M + \delta \log \frac{M}{B}}} = \frac{N}{M \cdot \left( \frac{M}{B} \right)^{\delta}}$$

of which all elements except one are in distinct blocks

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Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
Conclusion

\[ N / (M \cdot (\frac{M}{B})^{\Theta(\delta)}) \]

\[ \Theta(\log_\delta \frac{N}{M}) \]

\[ \frac{1}{\varepsilon} \log B \frac{N}{M} \]

\[ \frac{B^\varepsilon}{\varepsilon} \log B \frac{N}{M} \]

Buffered B-trees

B-trees

Insert

\[ \delta \]

Search

Sorting Threshold
Conclusion

\[ \frac{N}{M \cdot \left( \frac{M}{B} \right)^{\Theta(\delta)}} \]

\[ \Theta(\log_\delta \frac{N}{M}) \]

\[ \frac{1}{\varepsilon} \log_B \frac{N}{M} \]

\[ \log_B \frac{N}{M} \]

Search

Insert

Sorting Threshold

Buffered B-trees

B-trees

\[ \Theta(\log_{M/B} \frac{N}{M}) \]

\[ \log^{1+\varepsilon} N \]

\[ \frac{B^\varepsilon}{\varepsilon} \log_B \frac{N}{M} \]

\[ B/\log^3 N \]

\[ B \log_B \frac{N}{M} \]