Cache-Oblivious Dynamic Dictionaries with Update/Query Tradeoff

Gerth Stølting Brodal
Erik D. Demaine
Jeremy T. Fineman
John Iacono
Stefan Langerman
J. Ian Munro

Result presented at SODA 2010

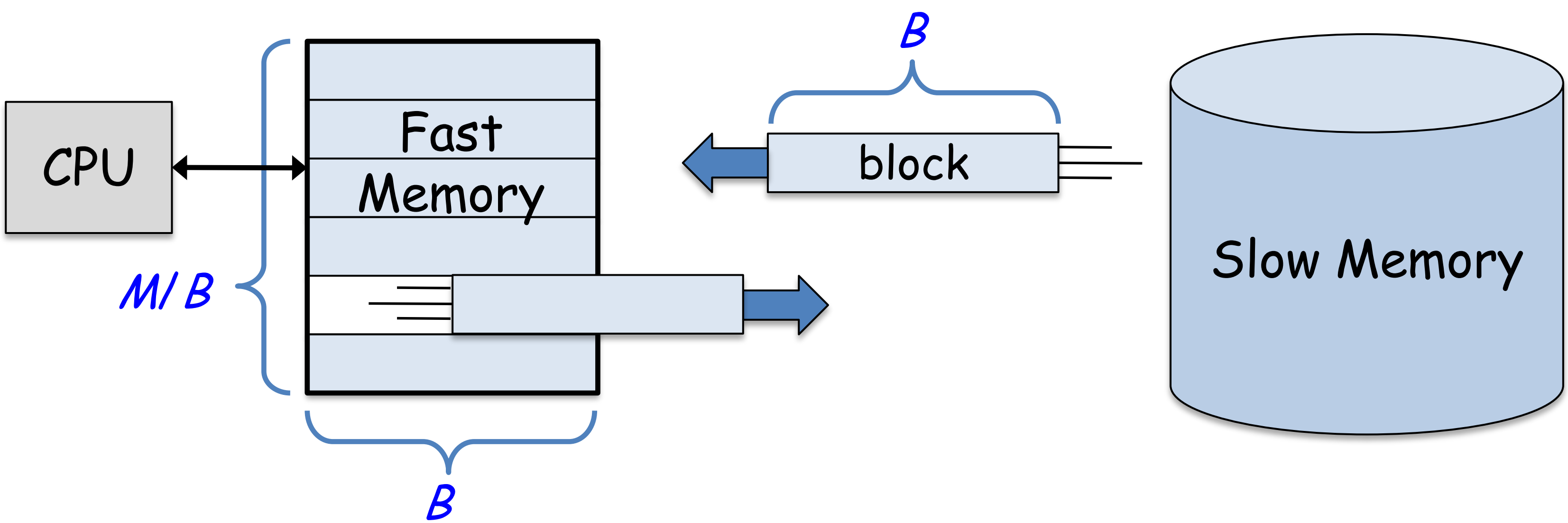
Dynamic Dictionary

Search(k)

Insert(e)
Delete(k)

I/O Model

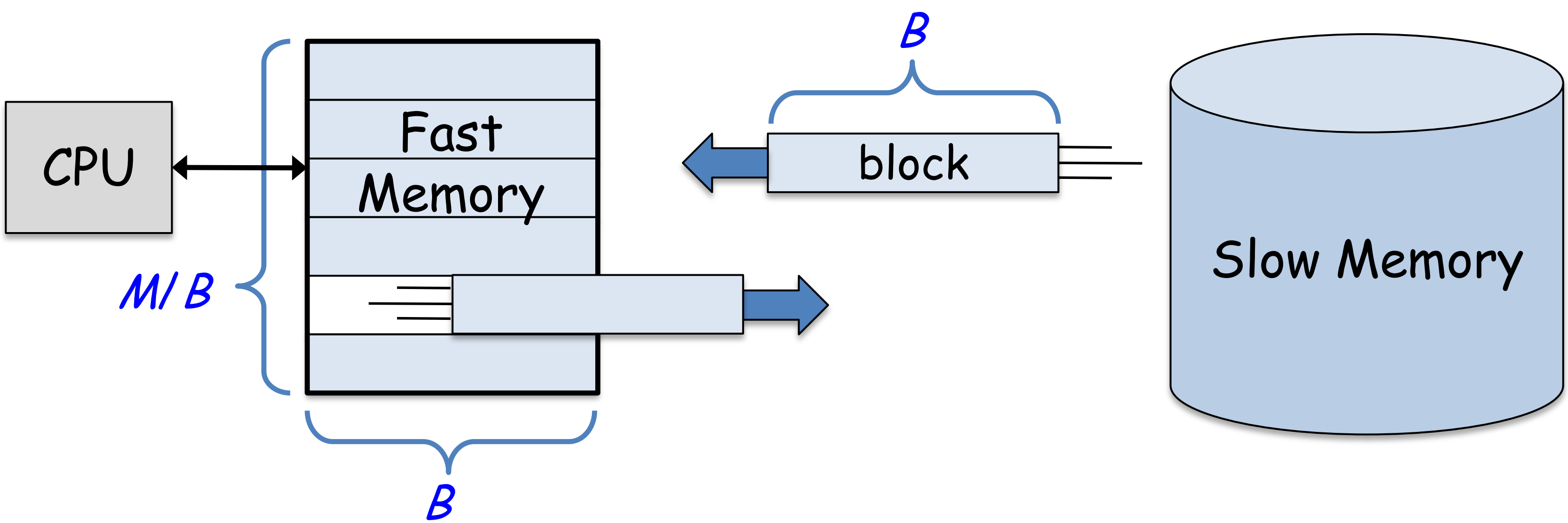
[Aggarwal, Vitter 88]



Cost: the number of ***block transfers*** (I/Os)

Cache-Oblivious Algorithms

[Frigo, Leiserson, Prokop, Ramachandran 99]



- Algorithms not parameterized by *M* or *B*
- Analyze in ***ideal-cache model*** — I/O model, except **optimal replacement** policy is assumed

Cache-Oblivious Dynamic Dictionaries

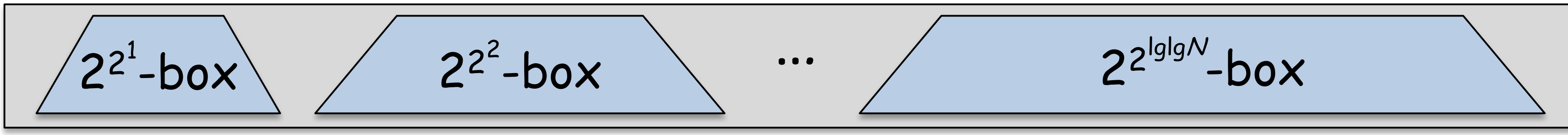
Cache-Aware	Search	Insert
B-tree [BM72]	$O(\log_B N)$	$O(\log_B N)$
Buffered B-tree [BF03]	$O((1/\varepsilon)\log_B N)$	$O((1/\varepsilon B^{1-\varepsilon})\log_B N)$*

Cache-Oblivious	Search	Insert
CO B-tree [BDF-00, BDIW04,BFJ02]	$O(\log_B N)$	$O(\log_B N + \ldots)$
COLA [BFF-CFKN07]	$O(\log_2 N)$	$O((1/B)\log_2 N)$*
Shuttle Tree [BFF-CFKN07]	$O(\log_B N)$	$O((1/B^{\Omega(1/(\log\log B)^2)})\log_B N + \ldots)$*
xDict [this paper]	$O((1/\varepsilon)\log_B N)$	$O((1/\varepsilon B^{1-\varepsilon})\log_B N)$*†

* amortized

† assumes $M = \Omega(B^2)$

Building an xDict (ε = 1/2)



$\lg\lg N$ **x-boxes** of squaring capacities

Insert: insert into smallest box

- When a box reaches capacity, **Flush** it and **Batch-Insert** into the next box
- $O((1/\sqrt{B})\log_B x)$ cost is dominated by largest box
  ➔ $O((1/\sqrt{B})\log_B N)$

Search: search in each x-box

- $O(\log_B x)$ cost is dominated by largest box $O(\log_B N)$

x-Box = **dictionary with capacity x^2**

Batch-Insert(D,A): insert $\Theta(x)$ presorted objects

— cost $O((1/\sqrt{B})\log_B x)$ per element

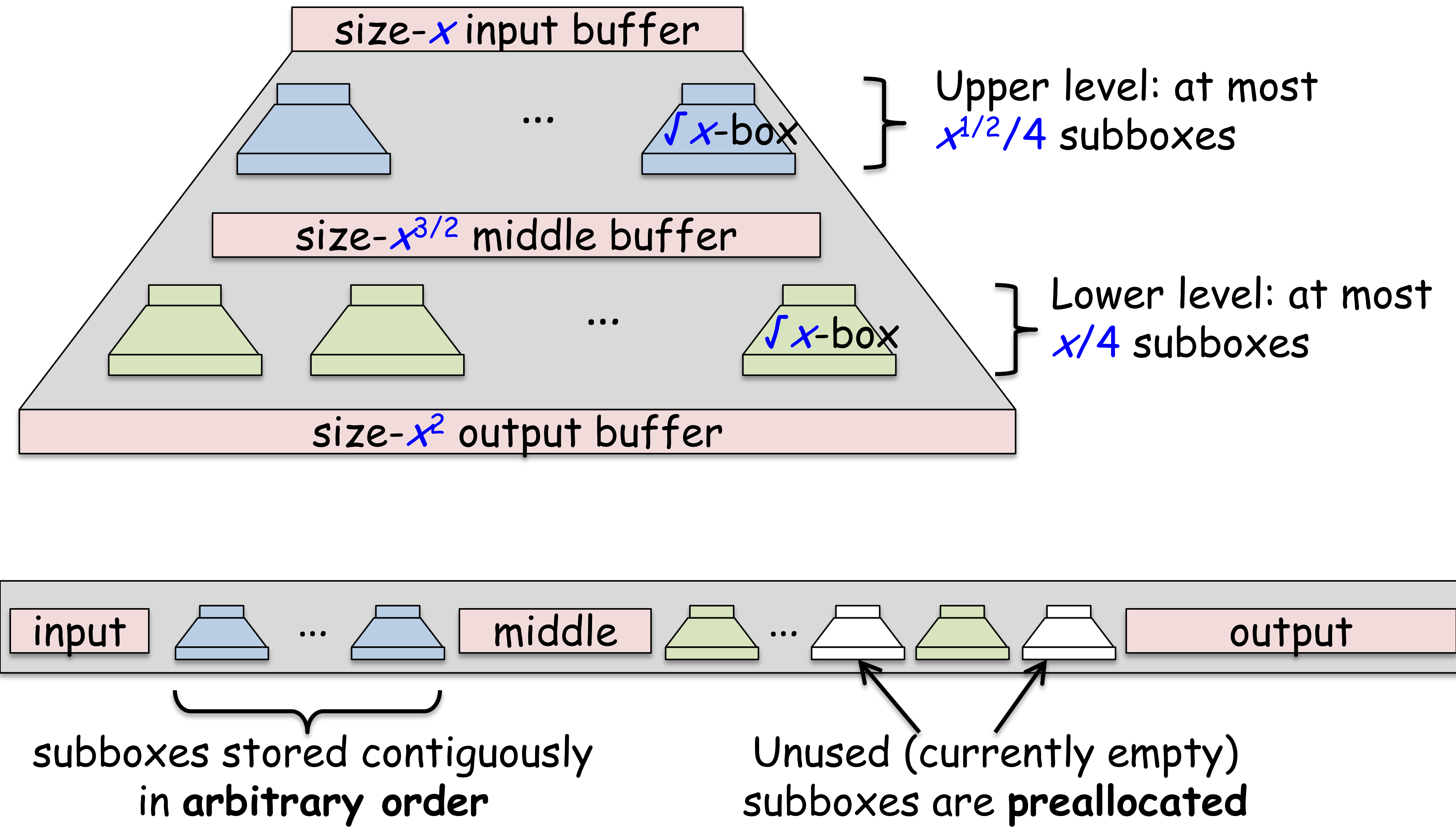
Search(D,K):

— cost is $O(\log_B x)$

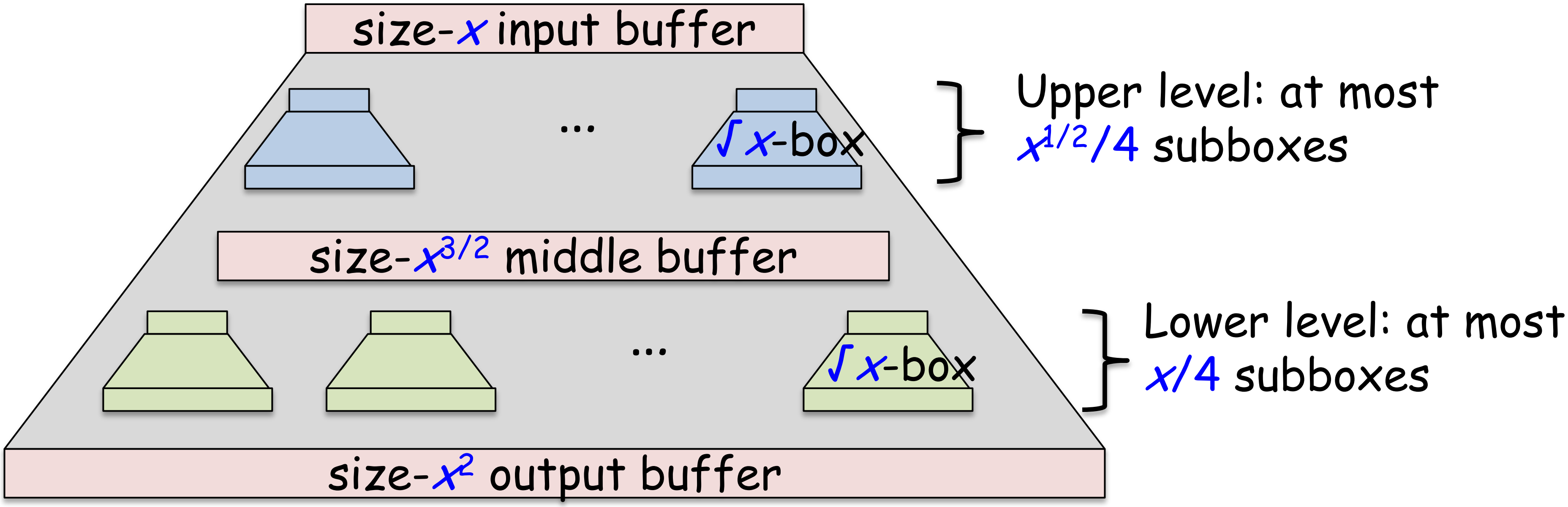
Flush(D): produce a size-x^2 sorted array A containing all the elements in the x-box D

— cost is $O(1/B)$ per element

Recursive x-Box



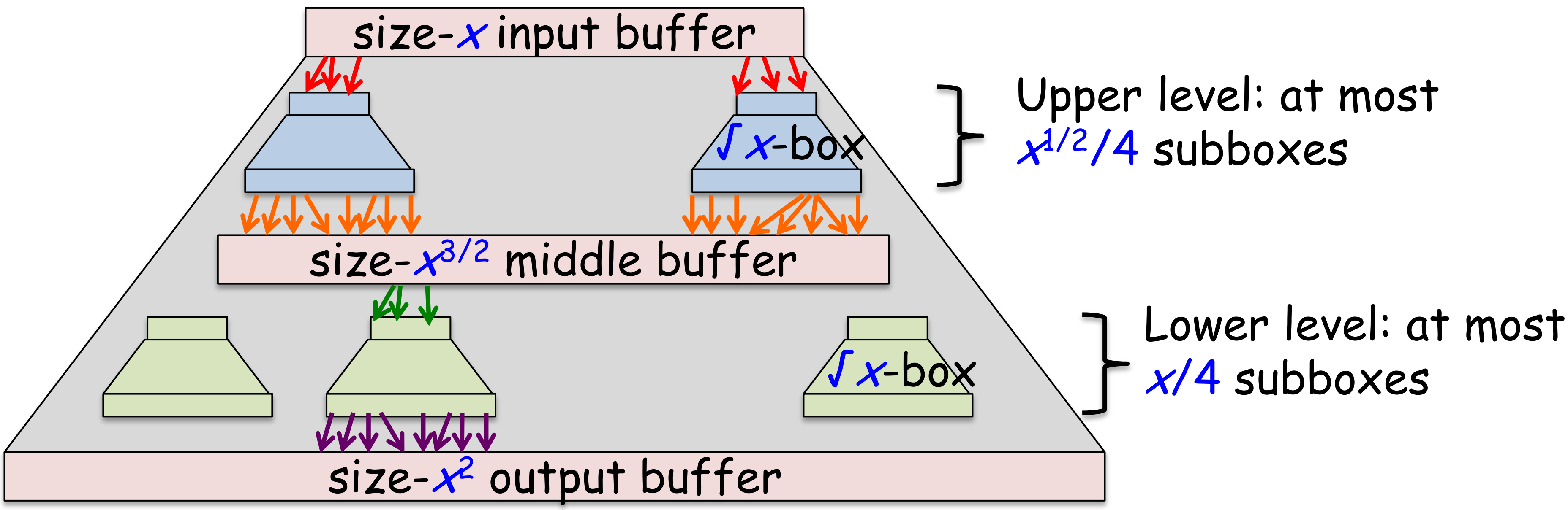
x-Box Space Usage



Theorem: An x-Box uses at most cx^2 space

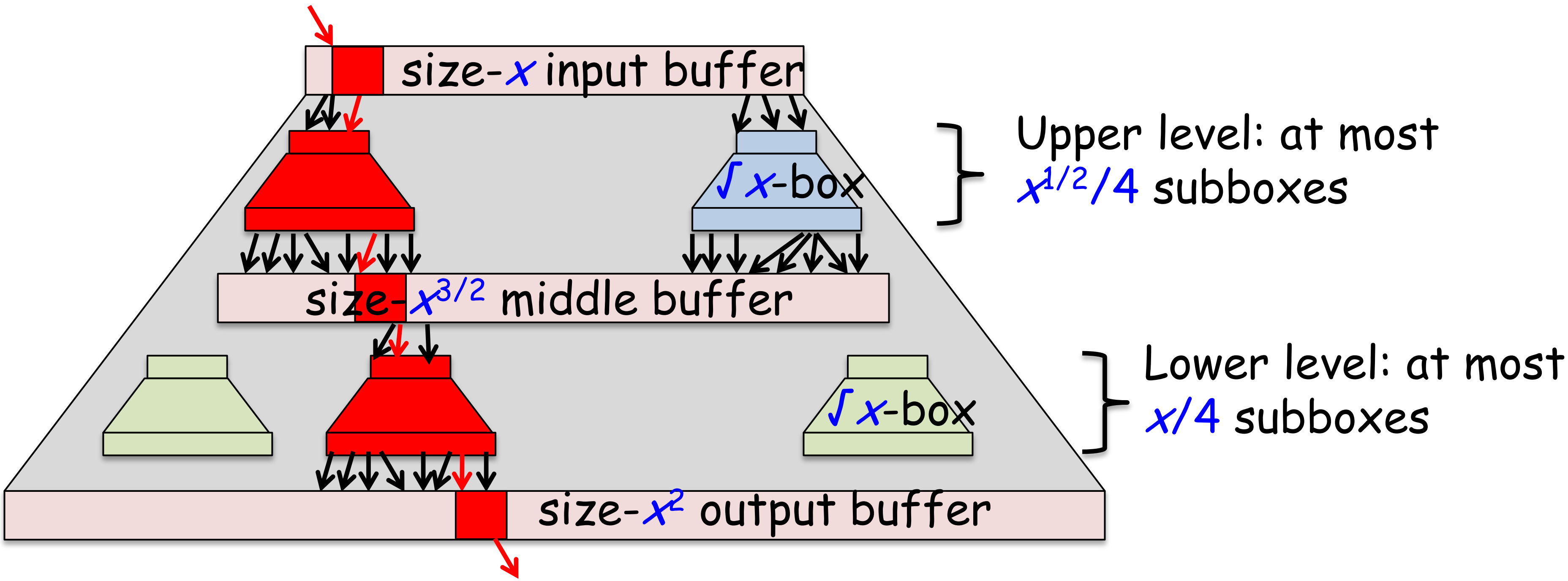
(within constant factor of capacity/output buffer)

Fractional Cascading within x-Box



Propagate samples upwards + Lookahead pointers

Searching in an x-Box



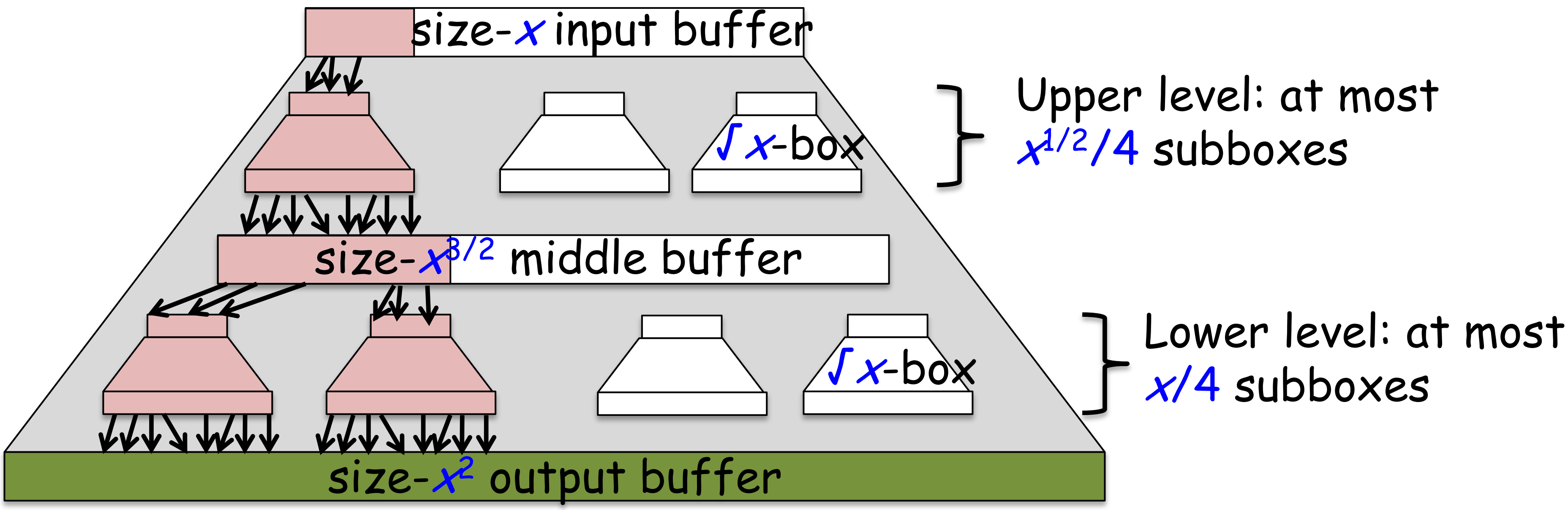
Describe searches by the recurrence

$$S(x) = 2S(\sqrt{x}) + O(1)$$

with base case $S(<\sqrt{B}) = 0$

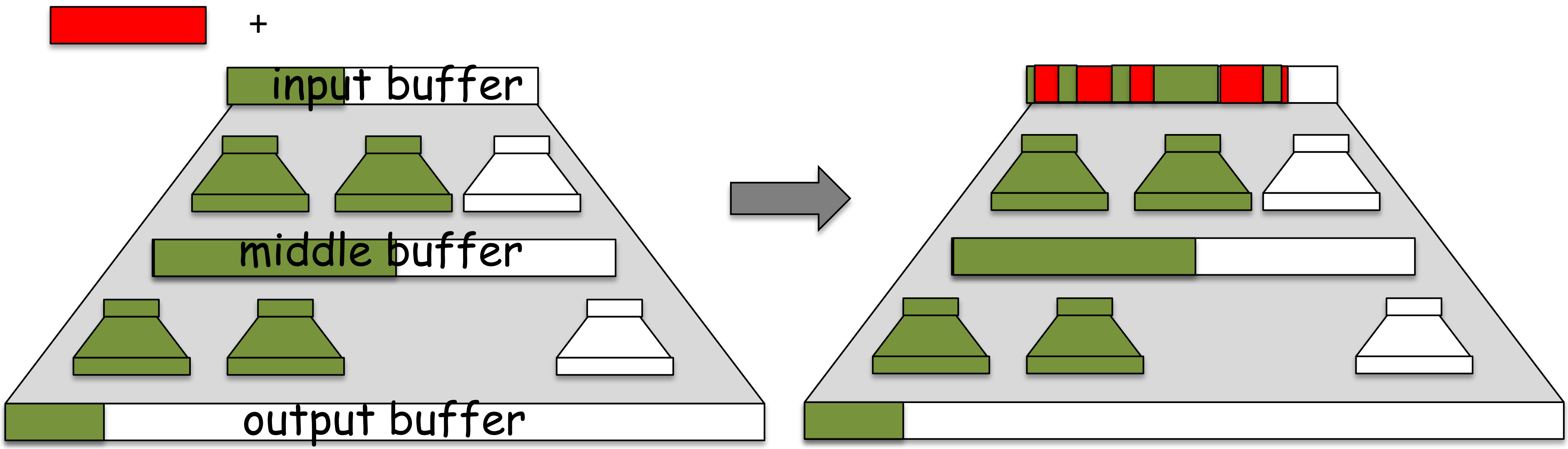
Solves to $O(\log_B N)$

Flush



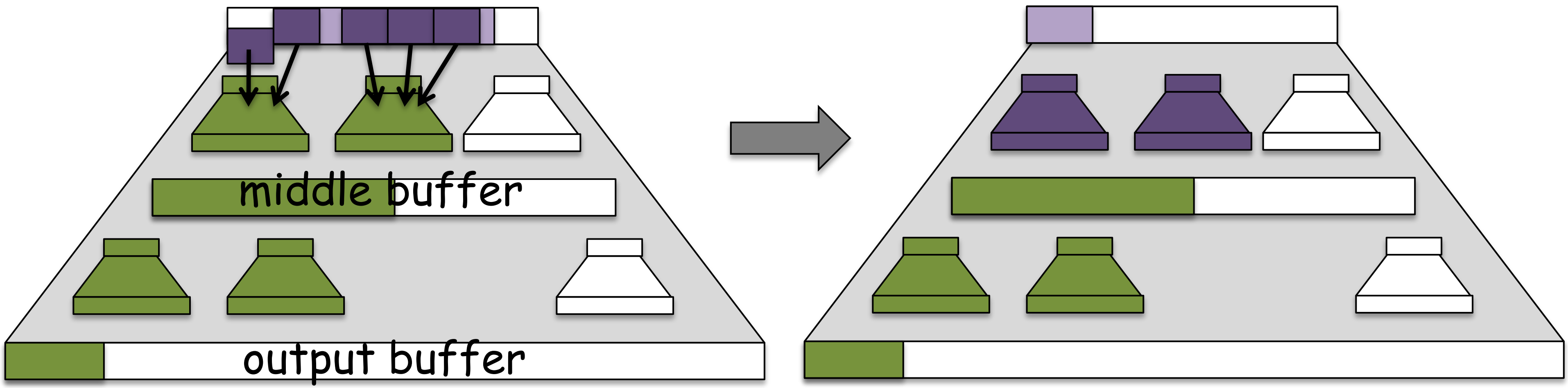
- Moves all real elements to the output buffer in sorted order.
- Lookahead pointers are rebuilt to facilitate searches. Most subboxes remain empty.

Batch-Insert



1. Merge sorted input into input buffer.

Batch-Insert



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2. If input buffer is "full enough," Batch-Insert into upper-level subboxes (in chunks of $\Theta(\sqrt{x})$)

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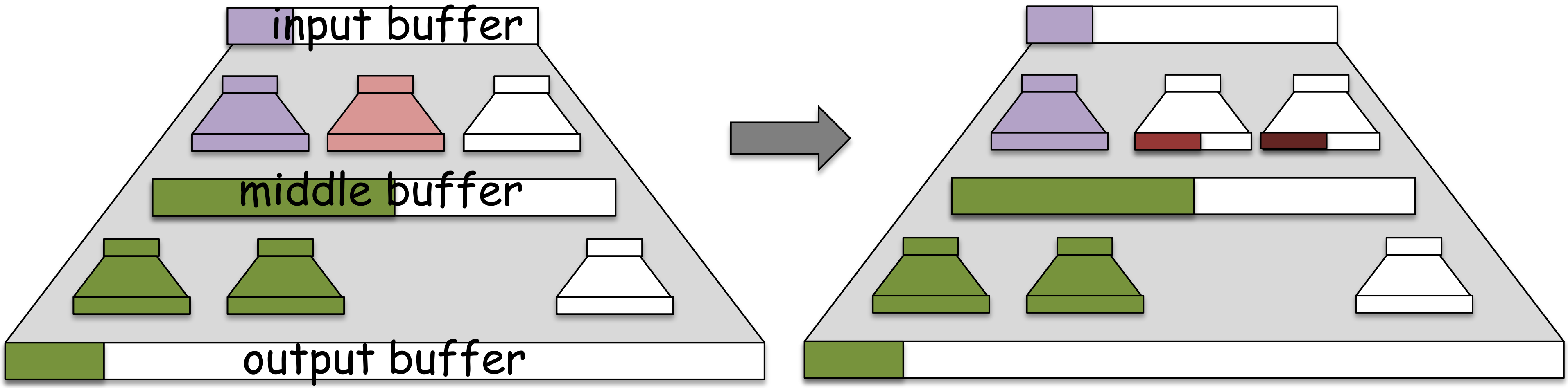
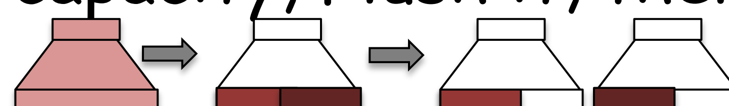


1. Merge sorted input into input buffer.
2. If input buffer is "full enough," Batch-Insert into upper-level subboxes (in chunks of $\Theta(\sqrt{x})$)
3. Whenever a subbox is near capacity, Flush it, then split it into two subboxes



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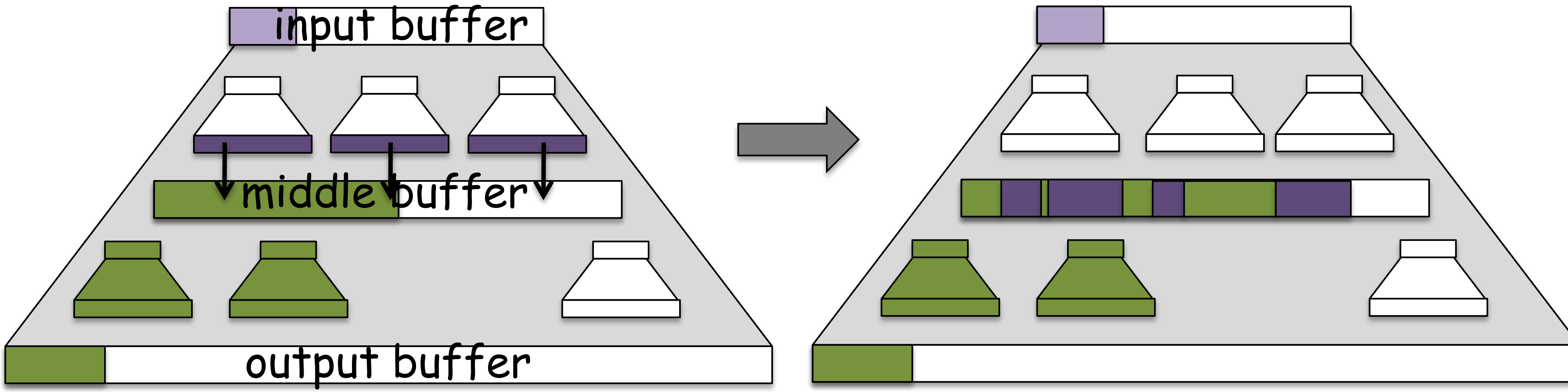

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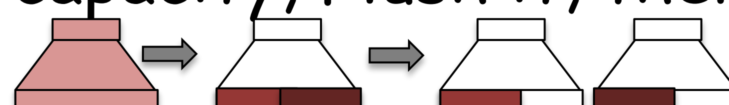
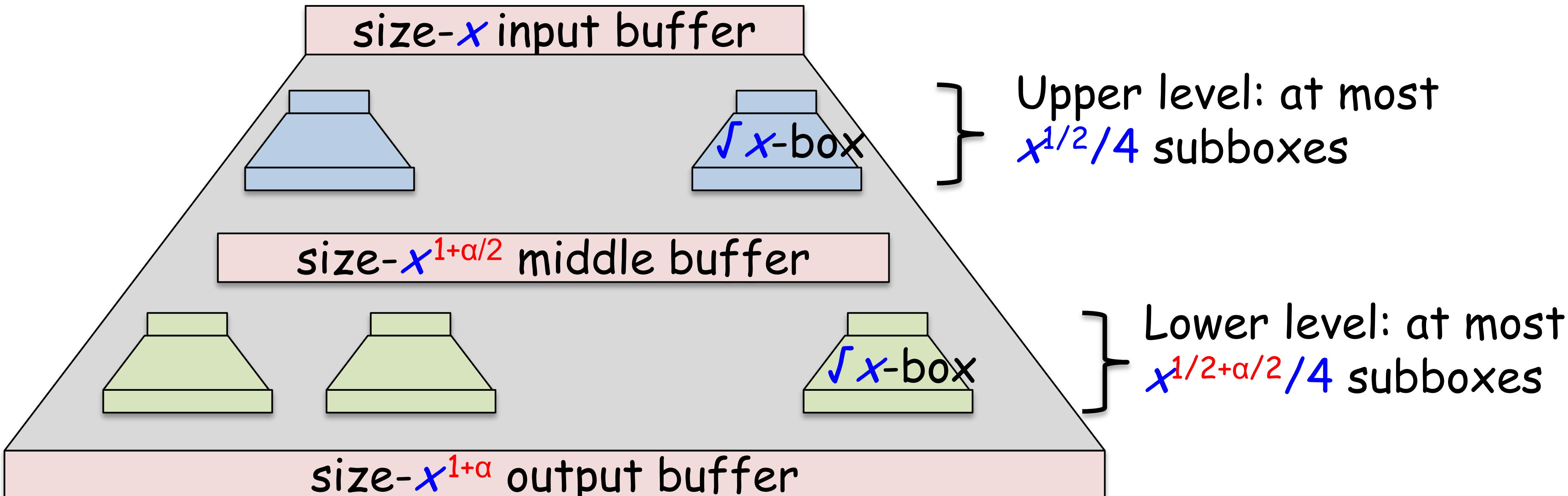


4. If no empty subboxes remain, Flush all of them and merge output buffers into middle buffer.

Generalizing to $O((1/\varepsilon B^{1-\varepsilon})\log_B N)$



Parameterize by $0 < \alpha \leq 1$, where $\alpha = \varepsilon/(1-\varepsilon)$

$1/\varepsilon$ overhead comes from geometric sum in xDict

Results Summary

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