

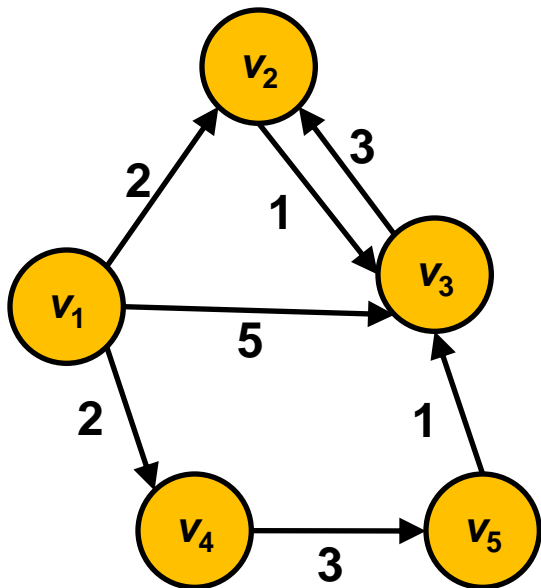
Algoritmer og Datastrukturer 2

Gerth Stølting Brodal

Korteste Veje
[CLRS, kapitel 25.1-25.2]



Korteste Veje mellem alle Par af Knude



d_{ij}

	1	2	3	4	5
1	0	2	3	2	5
2	$+\infty$	0	1	$+\infty$	$+\infty$
3	$+\infty$	3	0	$+\infty$	$+\infty$
4	$+\infty$	7	4	0	3
5	$+\infty$	4	1	$+\infty$	0

π_{ij}

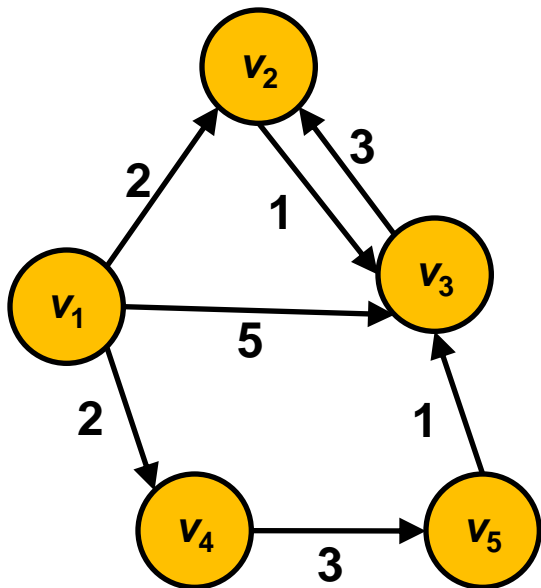
	1	2	3	4	5
1	NIL	1	2	1	4
2	NIL	NIL	2	NIL	NIL
3	NIL	3	NIL	NIL	NIL
4	NIL		5	NIL	4
5	NIL	3	5	NIL	NIL

PRINT-ALL-PAIRS-SHORTEST-PATH(Π, i, j)

```

1  if  $i == j$ 
2      print  $i$ 
3  elseif  $\pi_{ij} == \text{NIL}$ 
4      print “no path from”  $i$  “to”  $j$  “exists”
5  else PRINT-ALL-PAIRS-SHORTEST-PATH( $\Pi, i, \pi_{ij}$ )
6      print  $j$ 
    
```

Tid $O(n)$



d_{ij}

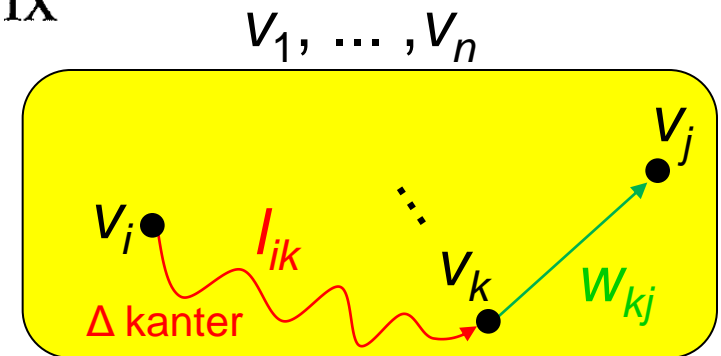
	1	2	3	4	5
1	0	2	3	2	5
2	$+\infty$	0	1	$+\infty$	$+\infty$
3	$+\infty$	3	0	$+\infty$	$+\infty$
4	$+\infty$	7	4	0	3
5	$+\infty$	4	1	$+\infty$	0

Π_{ij}

	1	2	3	4	5
1	NIL	1	2	1	4
2	NIL	NIL	2	NIL	NIL
3	NIL	3	NIL	NIL	NIL
4	NIL	3	5	NIL	4
5	NIL	3	5	NIL	NIL

EXTEND-SHORTEST-PATHS (L, W)

```
1   $n = L.rows$ 
2  let  $L' = (l'_{ij})$  be a new  $n \times n$  matrix
3  for  $i = 1$  to  $n$ 
4      for  $j = 1$  to  $n$ 
5           $l'_{ij} = \infty$ 
6          for  $k = 1$  to  $n$ 
7               $l'_{ij} = \min(l'_{ij}, l_{ik} + w_{kj})$ 
8  return  $L'$ 
```



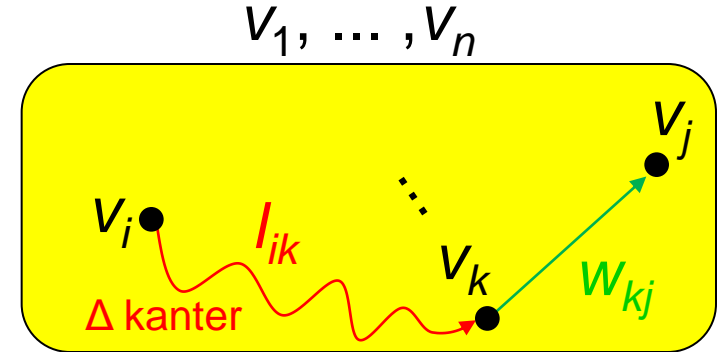
L_{ij} = korteste afstand fra i til j for stier med Δ kanter

W = incidensmatricen

L'_{ij} = korteste afstand fra i til j for stier med $\Delta+1$ kanter

SQUARE-MATRIX-MULTIPLY(A, B)

```
1   $n = A.rows$ 
2  let  $C$  be a new  $n \times n$  matrix
3  for  $i = 1$  to  $n$ 
4      for  $j = 1$  to  $n$ 
5           $c_{ij} = 0$ 
6          for  $k = 1$  to  $n$ 
7               $c_{ij} = c_{ij} + a_{ik} \cdot b_{kj}$ 
8  return  $C$ 
```

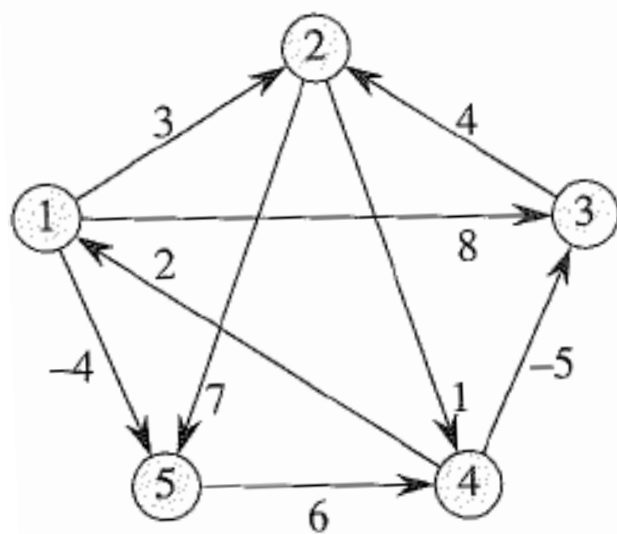


SLOW-ALL-PAIRS-SHORTEST-PATHS (W)

- 1 $n = W.rows$
- 2 $L^{(1)} = W$ ← diagonalen = 0
- 3 **for** $m = 2$ **to** $n - 1$
- 4 let $L^{(m)}$ be a new $n \times n$ matrix
- 5 $L^{(m)} = \text{EXTEND-SHORTEST-PATHS}(L^{(m-1)}, W)$ → →
- 6 **return** $L^{(n-1)}$

$L^{(m)}_{ij}$ = korteste afstand fra i til j for stier med m kanter
 W = incidensmatricen

Tid $O(n^4)$

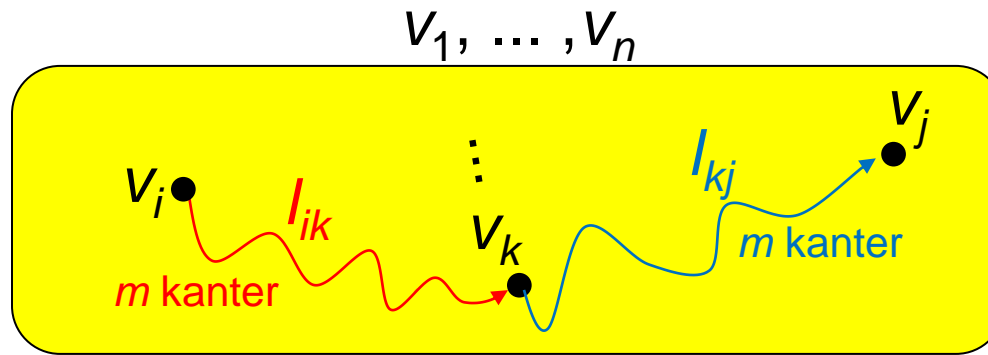


$$L^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$L^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{pmatrix}$$

$$L^{(3)} = \begin{pmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

$$L^{(4)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$



FASTER-ALL-PAIRS-SHORTEST-PATHS(W)

- 1 $n = W.rows$
- 2 $L^{(1)} = W$
- 3 $m = 1$
- 4 **while** $m < n - 1$
- 5 let $L^{(2m)}$ be a new $n \times n$ matrix
- 6 $L^{(2m)} = \text{EXTEND-SHORTEST-PATHS}(L^{(m)}, L^{(m)})$
- 7 $m = 2m$
- 8 **return** $L^{(m)}$

$L^{(m)}_{ij}$ = korteste afstand fra i til j for stier med m kanter
 W = incidensmatricen

Tid $O(n^3 \cdot \log n)$

Floyd-Warshall

FLOYD-WARSHALL(W)

1 $n = W.rows$

2 $D^{(0)} = W$

3 **for** $k = 1$ **to** n

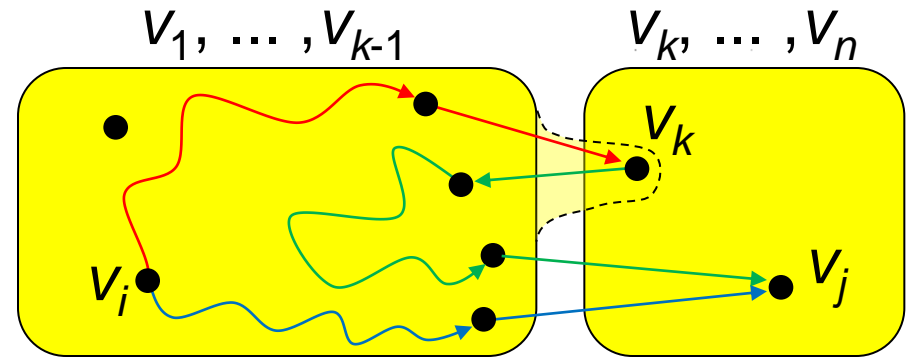
4 let $D^{(k)} = (d_{ij}^{(k)})$ be a new $n \times n$ matrix

5 **for** $i = 1$ **to** n

6 **for** $j = 1$ **to** n

7 $d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$

8 **return** $D^{(n)}$



$d_{ij}^{(k)}$ = korteste vej fra i til j der kun går **via** $1..k$ Tid $O(n^3)$

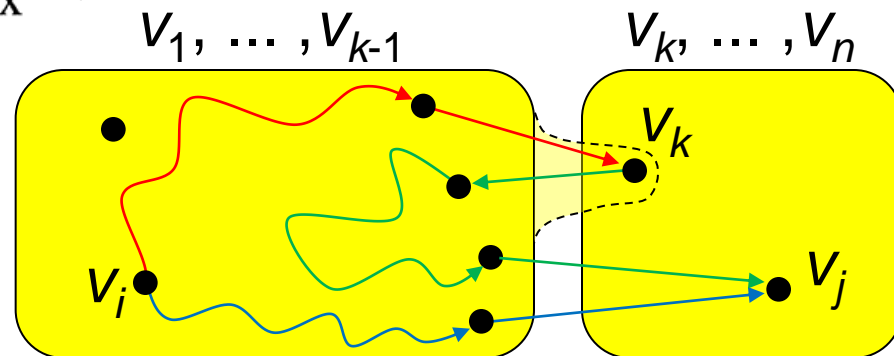
Transitive Lukning (= Floyd-Warshall simplificeret)

TRANSITIVE-CLOSURE(G)

```

1   $n = |G.V|$ 
2  let  $T^{(0)} = (t_{ij}^{(0)})$  be a new  $n \times n$  matrix
3  for  $i = 1$  to  $n$ 
4    for  $j = 1$  to  $n$ 
5      if  $i == j$  or  $(i, j) \in G.E$ 
6         $t_{ij}^{(0)} = 1$ 
7      else  $t_{ij}^{(0)} = 0$ 
8  for  $k = 1$  to  $n$ 
9    let  $T^{(k)} = (t_{ij}^{(k)})$  be a new  $n \times n$  matrix
10   for  $i = 1$  to  $n$ 
11     for  $j = 1$  to  $n$ 
12        $t_{ij}^{(k)} = t_{ij}^{(k-1)} \vee (t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)})$ 
13  return  $T^{(n)}$ 

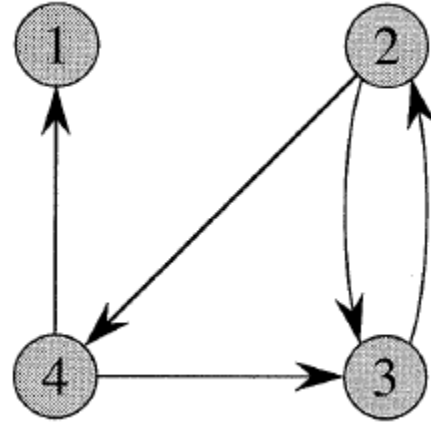
```



$t_{ij}^{(k)}$ = findes en vej fra i til j der kun går via $1..k$

Tid $O(n^3)$

Transitive Lukning: Eksempel



$$T^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

$$T^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

$$T^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

$$T^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$T^{(4)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Opsummering Korteste Veje

		SSSP En-til-alle korteste veje	APSP Alle-til-alle korteste veje
Acykliske grafer (positive og negative vægte)		$O(n+m)$	$O(n \cdot (n+m))$
Generelle grafer	Kun positive vægte	Dijkstra $O((n+m) \cdot \log n)$ $O(n^2+m)$	Floyd-Warshall $O(n^3)$
	Positive og negative vægte	Bellman-Ford $O(m \cdot n)$	