Algoritmer og Datastrukturer 2

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Korteste Veje
[CLRS, kapitel 25.1-25.2]

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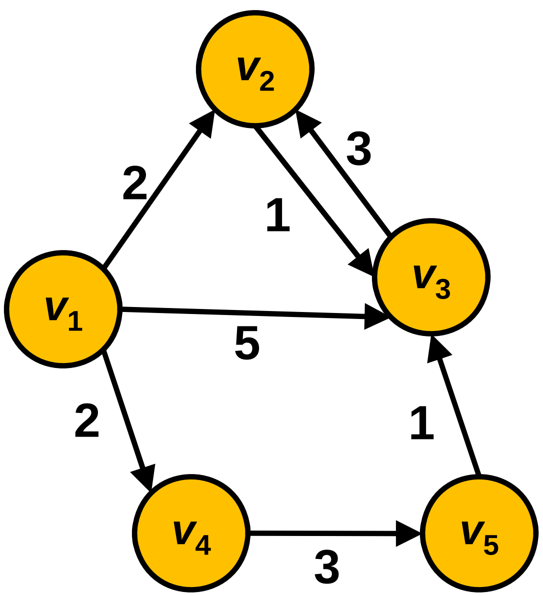
Korteste Veje mellem alle Par af Knude



d_{ij}

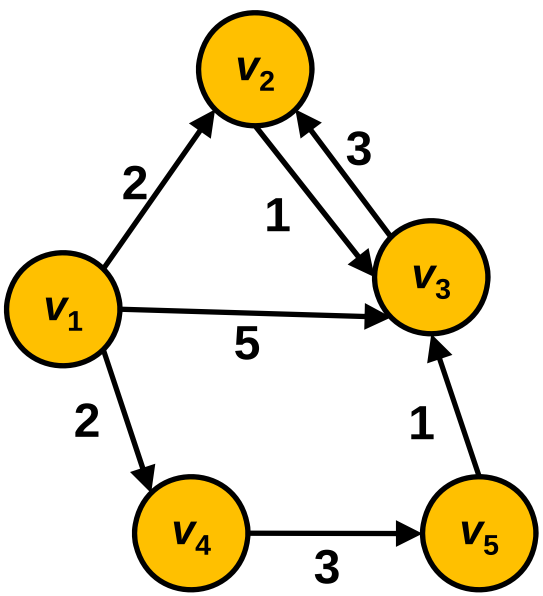
	1	2	3	4	5
1	0	2	3	2	5
2	$+\infty$	0	1	$+\infty$	$+\infty$
3	$+\infty$	3	0	$+\infty$	$+\infty$
4	$+\infty$	7	4	0	3
5	$+\infty$	4	1	$+\infty$	0

π_{ij}

	1	2	3	4	5
1	NIL	1	2	1	4
2	NIL	NIL	2	NIL	NIL
3	NIL	3	NIL	NIL	NIL
4	NIL		5	NIL	3
5	NIL	3	5	NIL	NIL

PRINT-ALL-PAIRS-SHORTEST-PATH(Π, i, j)

```
if i == j
    print i
elseif π_ij == NIL
    print "no path from" i "to" j "exists"
else PRINT-ALL-PAIRS-SHORTEST-PATH(Π, i, π_ij)
    print j
```



d_{ij}

	1	2	3	4	5
1	0	2	3	2	5
2	+∞	0	1	+∞	+∞
3	+∞	3	0	+∞	+∞
4	+∞	7	4	0	3
5	+∞	4	1	+∞	0

π_{ij}

	1	2	3	4	5
1	NIL	1	2	1	4
2	NIL	NIL	2	NIL	NIL
3	NIL	3	NIL	NIL	NIL
4	NIL	3	5	NIL	4
5	NIL	3	5	NIL	NIL

EXTEND-SHORTEST-PATHS(L, W)

```
n = L.rows
let L' = (l'_ij) be a new n × n matrix
for i = 1 to n
    for j = 1 to n
        l'_ij = ∞
        for k = 1 to n
            l'_ij = min(l'_ij, l_ik + w_kj)
return L'
```

L_{ij} = korteste afstand fra i til j for stier med Δ kanter
W = incidensmatricen
L'_{ij} = korteste afstand fra i til j for stier med Δ+1 kanter

Tid $O(n^3)$

```
SQUARE-MATRIX-MULTIPLY(A, B)
n = A.rows
let C be a new n × n matrix
for i = 1 to n
    for j = 1 to n
        c_ij = 0
        for k = 1 to n
            c_ij = c_ij + a_ik · b_kj
return C
```

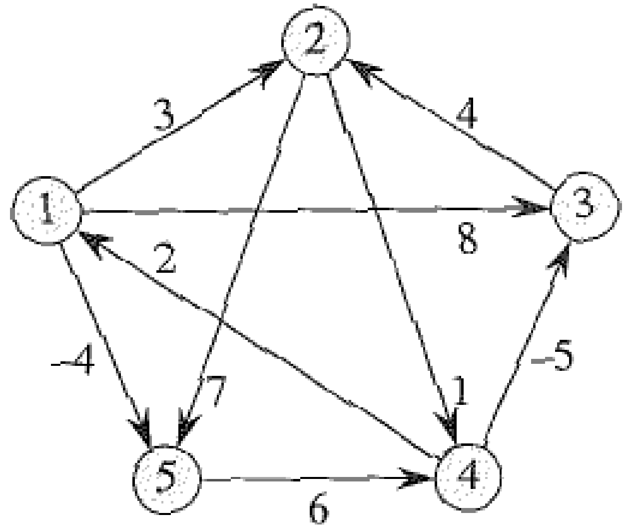
Tid $O(n^3)$

SLOW-ALL-PAIRS-SHORTEST-PATHS(W)

$n = W.rows$
$L^{(1)} = W$ ← diagonalen = 0
for $m = 2$ **to** $n - 1$
    let $L^{(m)}$ be a new $n \times n$ matrix
    $L^{(m)} =$ EXTEND-SHORTEST-PATHS$(L^{(m-1)}, W)$
return $L^{(n-1)}$

$L^{(m)}_{ij}$ = korteste afstand fra *i* til *j* for stier med m kanter
W = incidensmatricen

Tid $O(n^4)$



$$L^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad L^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{pmatrix}$$

$$L^{(3)} = \begin{pmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \quad L^{(4)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

FASTER-ALL-PAIRS-SHORTEST-PATHS(W)

$n = W.rows$

$L^{(1)} = W$

$m = 1$

while $m < n - 1$

    let $L^{(2m)}$ be a new $n \times n$ matrix

    $L^{(2m)} =$ EXTEND-SHORTEST-PATHS$(L^{(m)}, L^{(m)})$

    $m = 2m$

return $L^{(m)}$

$L^{(m)}_{ij}$ = korteste afstand fra *i* til *j* for stier med m kanter

W = incidensmatricen

Tid $O(n^3 \cdot \log n)$

Floyd-Warshall

FLOYD-WARSHALL(W)

$n = W.rows$
$D^{(0)} = W$
for $k = 1$ **to** n
    let $D^{(k)} = \left(d_{ij}^{(k)}\right)$ be a new $n \times n$ matrix
    for $i = 1$ **to** n
        for $j = 1$ **to** n
            $d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$
return $D^{(n)}$

$d^{(k)}_{ij}$ = korteste vej fra i til j der kun går via $1..k$

Tid $O(n^3)$

Transitive Lukning

(= Floyd-Warshall simplificeret)

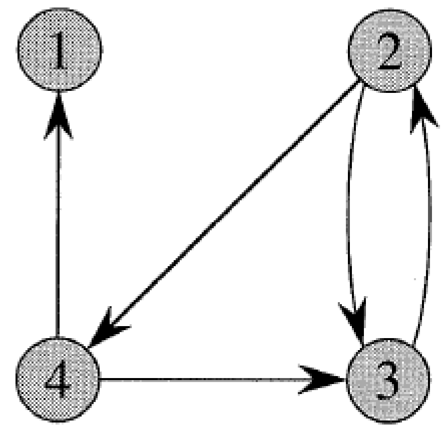
TRANSITIVE-CLOSURE(G)

$n = |G.V|$
let $T^{(0)} = \left(t_{ij}^{(0)}\right)$ be a new $n \times n$ matrix
for $i = 1$ **to** n
    for $j = 1$ **to** n
        if $i == j$ or $(i, j) \in G.E$
            $t_{ij}^{(0)} = 1$
        else $t_{ij}^{(0)} = 0$
for $k = 1$ **to** n
    let $T^{(k)} = \left(t_{ij}^{(k)}\right)$ be a new $n \times n$ matrix
    for $i = 1$ **to** n
        for $j = 1$ **to** n
            $t_{ij}^{(k)} = t_{ij}^{(k-1)} \vee \left(t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)}\right)$
return $T^{(n)}$

$t^{(k)}_{ij}$ = findes en vej fra i til j der kun går via $1..k$

Tid $O(n^3)$

Transitive Lukning: Eksempel



$$T^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad T^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad T^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

$$T^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad T^{(4)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Opsummering

		SSSP En-til-alle korteste veje	APSP Alle-til-alle korteste veje
Acykliske grafer (positive og negative vægte)		$O(n+m)$	$O(n\cdot(n+m))$
Generelle grafer	Kun positive vægte	**Dijkstra** $O((n+m)\cdot\log n)$	**Floyd-Warshall** $O(n^3)$
	Positive og negative vægte	**Bellman-Ford** $O(m\cdot n)$	