Homework Assignment 2
Solution

Q2: **(a,b)-trees**

Assume in the following all trees have at least 2 leaves.

- To prove that insertions and deletions in a (2,4)-tree require amortized $O(1)$ node splittings, fusions and sharings, we use the following accounting:

  Each node of degree 2 has 1 credit, and each node of degree 4 has 2 credits.

  Each credit can pay for one node splitting, fusion or sharing.

  Assume an insertion performs $k \geq 0$ node splittings. Then k nodes of degree 4 (releasing $2k$ credits) are replaced by k nodes of degree 2 (requiring k credits) and k nodes of degree 3 (requiring 0 credits), and one node of degree $2 \leq d < 4$ is replaced by one node of degree $d+1$ (releasing 1 credit if $d = 2$, and requiring 2 credits if $d = 3$) or a new root is created (requiring 1 credit). The total change in required credit is at most $-2k + k + 2 \leq -k + 2$. The amortized cost of an insertion becomes at most $k + (-k + 2) \leq 2$.

  Assume a deletion performs $k \geq 0$ node fusions. Then $2k$ nodes of degree 2 (releasing $2k$ credits) are replaced by k nodes of degree 3 (requiring 0 credits). If the root is deleted one degree 2 node is deleted (releasing 1 credit). If no sharing is required one node of degree $2 < d \leq 4$ is replaced by one node of degree $d-1$ (releasing 2 credits if $d = 4$, and requiring 1 credit if $d = 3$). If a sharing is required one node of degree $2 < d \leq 4$ is again replaced by one node of degree $d-1$. The total change of required credit is at most $-2k+1$. The amortized cost of a deletion becomes at most $(-2k+1) + (k+1) \leq 2$.

  The amortized $O(1)$ restructuring cost of $(2,4)$-trees follows.

- The restructuring of an (a,b)-tree is done as follows. After a leaf creation, repeatedly split nodes of degree $b+1$ into two nodes of degree $\lceil \frac{b+1}{2} \rceil$ and $\lfloor \frac{b+1}{2} \rfloor$. After a leaf deletion, repeatedly do the following restructuring: 1) if the root has degree 1 then delete the root, 2) if a node has degree $a-1$ and has a sibling of degree a fusion the two nodes to a degree $2a-1$ node, and 3) if a node has degree $a-1$ and has a sibling of degree $d > a$ then move one child of the degree d node to the degree $a-1$ node.

  Let $\epsilon > 0$ be a constant, and $b \geq (2+\epsilon)a$. Observe:

  $$a < \left\lfloor \frac{b+1}{2} \right\rfloor \leq \left\lceil \frac{b+1}{2} \right\rceil < b \quad \text{and} \quad a < 2a - 1 < b$$

  Let $\alpha = \min\left\{2a-1, \lfloor \frac{b+1}{2} \rfloor\right\}$ and $\beta = \max\left\{2a-1, \lceil \frac{b+1}{2} \rceil\right\}$. Then $a < \alpha \leq \beta < b$, and the resulting nodes of splitting a node and fusion two nodes have degree in $[\alpha, \beta]$.

  The following accounting will be used to argue about the amortized number of edges moved. Each credit can pay for moving one edge.

  Each node v has the number of credits given by the following expression, where d is the degree of v:

  $$\begin{cases} \left\lceil \frac{a}{2} \frac{\alpha - d}{\alpha - a} \right\rceil & \text{if } d \leq \alpha \text{ and } v \neq \text{the root}, \\ \left\lfloor \frac{b+1}{2} \frac{d-\beta}{b-\beta} \right\rfloor & \text{if } \beta \leq d \leq b, \\ 0 & \text{otherwise}. \end{cases}$$

  Nodes with degree in $[\alpha, \beta]$ do not have credits associated, i.e., splitting a node and fusion two nodes result in nodes with no credit. Nodes of degree a have $\lceil \frac{a}{2} \rceil$ credits and nodes of degree b have $\lfloor \frac{b+1}{2} \rfloor$ credits.

  Consider an insertion performing $k \geq 0$ splittings. Then k nodes of degree b are split (releasing $k \lfloor \frac{b+1}{2} \rfloor$ credits) into k nodes of degree $\lfloor \frac{b+1}{2} \rfloor$ (requiring 0 credits) and k nodes of degree $\lceil \frac{b+1}{2} \rceil$ (requiring 0 credits). If the last splitting creates a new root, the new root requires 0 credits.

Otherwise the degree of an existing node v increases by one. Let d denote the degree of v. If $d < \beta$, the required credit at v only can decrease. If $d \geq \beta$, the required credit at the node increases by

$$\left\lfloor \frac{b+1}{2}\frac{(d+1)-\beta}{b-\beta} \right\rfloor - \left\lfloor \frac{b+1}{2}\frac{d-\beta}{b-\beta} \right\rfloor \leq \frac{b+1}{2}\frac{(d+1)-\beta}{b-\beta} - \frac{b+1}{2}\frac{d-\beta}{b-\beta} + 1 \leq \frac{b+1}{2(b-\beta)} + 1 \; .$$

In total the splittings move $k\left\lfloor \frac{b+1}{2} \right\rfloor$ edges, and the amortized number of edges moved by the insertion becomes at most

$$k\left\lfloor \frac{b+1}{2} \right\rfloor + \left(-k\left\lfloor \frac{b+1}{2} \right\rfloor + \frac{b+1}{2(b-\beta)} + 1\right) = \frac{b+1}{2(b-\beta)} + 1 \; .$$

Consider a deletion performing k node fusions. Then $2k$ nodes of degree a (releasing $2k\left\lceil \frac{a}{2} \right\rceil \geq ka$ credits) are replaced by k nodes of degree $2a-1$ (requiring 0 credits). Furthermore either the root is deleted, or one existing node with degree $d > a$ gets degree $d-1$. If $d > \alpha$ the required credit at the node only can decrease. If $a < d \leq \alpha$ the required credit at the node increases by

$$\left\lceil \frac{a}{2}\frac{\alpha-(d-1)}{\alpha-a} \right\rceil - \left\lceil \frac{a}{2}\frac{\alpha-d}{\alpha-a} \right\rceil \leq \frac{a}{2}\frac{\alpha-(d-1)}{\alpha-a} - \frac{a}{2}\frac{\alpha-d}{\alpha-a} + 1 \leq \frac{a}{2(\alpha-a)} + 1 \; .$$

In total at most $ka+1$ edges are moved, and the amortized number of edges moved by the deletion becomes at most

$$(ka+1) + (-ka + \frac{a}{2(\alpha-a)} + 1) \leq \frac{a}{2(\alpha-a)} + 2 \; .$$

Be calculation it follows that

$$\max\left\{ \frac{b+1}{2(b-\beta)} + 1, \frac{a}{2(\alpha-a)} + 2 \right\} \leq O\left(\frac{1}{\epsilon}\right) \; ,$$

independent of a and b. It follows that insertions and deletions move amortized $O(1/\epsilon)$ edges, which is $O(1)$ for $\epsilon > 0$ constant.

– If $b = 2a$ construct a tree with $2a-1$ leaves. Now alternate between inserting two leaves and deleting two leaves. After two insertions there are $b+1$ leaves, and the tree must have at least two internal nodes. After two deletions the tree has $2a-1$ leaves and has only one internal node. It follows that after every second operation at least a edges must have been moved or deleted, i.e., on average each operation moves $\Omega(a)$ edges.