Dynamic Planar Convex Hull

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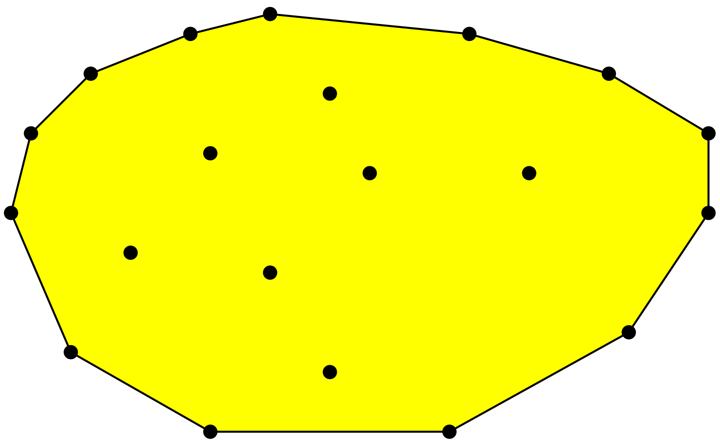
Outline of the talk

- Planar convex hull
- Duality: Lower Envelope
- Application: k-level
- Overall structure of the data structure
- Some key ingredients
- Lower bounds

Planar Convex Hull

Input A set of points $S \subseteq \mathbb{R}^2$

Output The points on the convex hull $\mathrm{CH}(S)$ in clockwise order



$n = |S|$ $\qquad$ $h = |\mathrm{CH}(S)|$

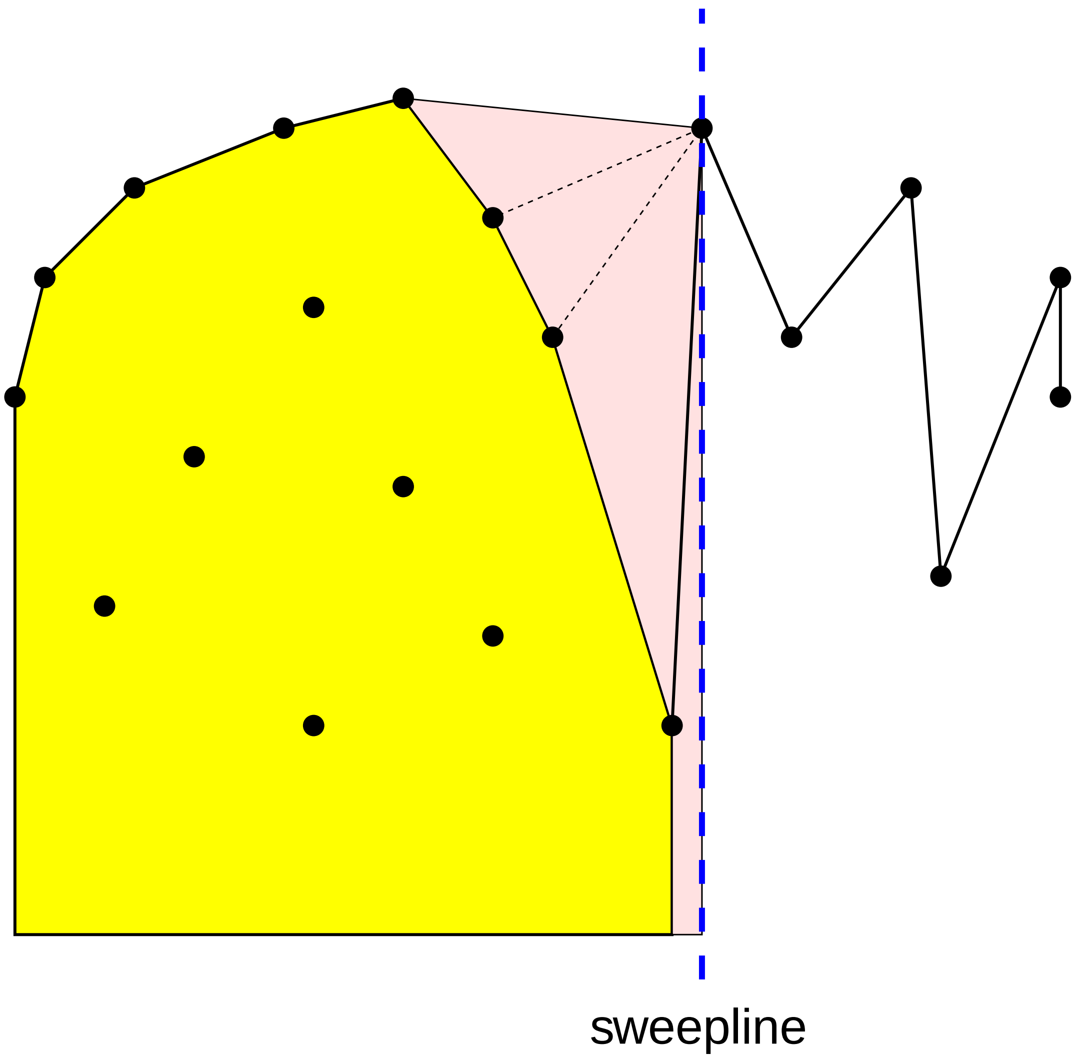
Known results

Optimal $O(n \log n)$ Graham 1972; ...

Output-sensitive $O(n \log h)$ Kirkpatrick, Seidel 1986; Chan 1996

Graham's Scan

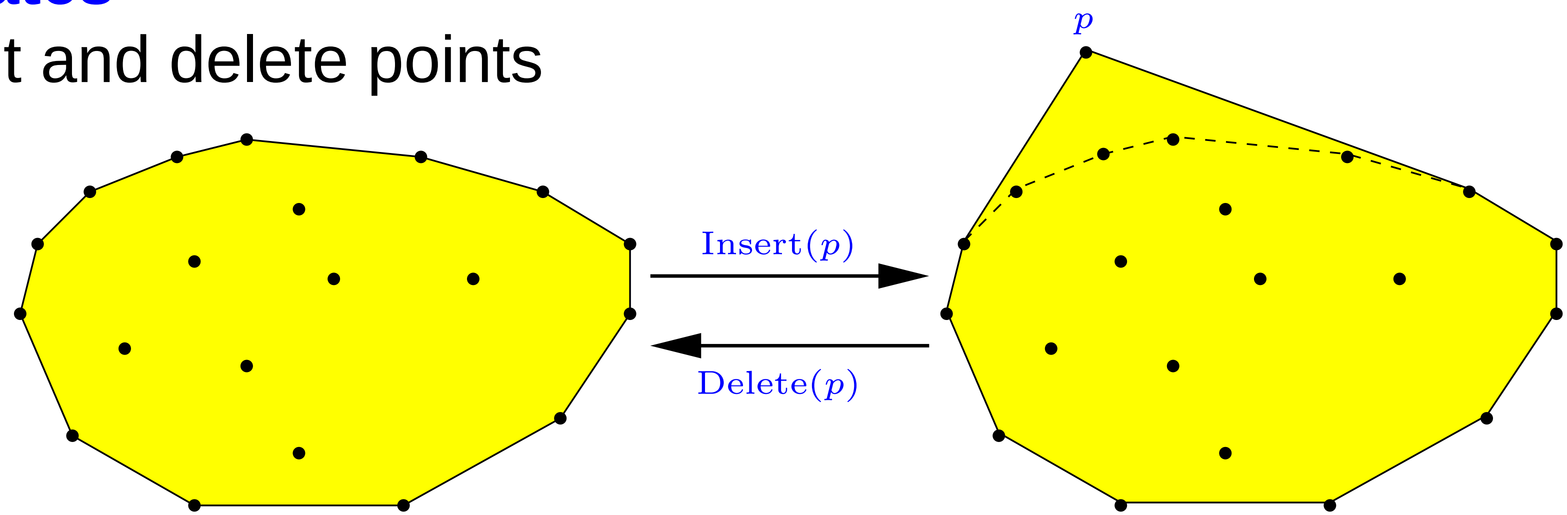
Andrew's variant for upper hull



Dynamic Planar Convex Hull

Updates

Insert and delete points



Queries

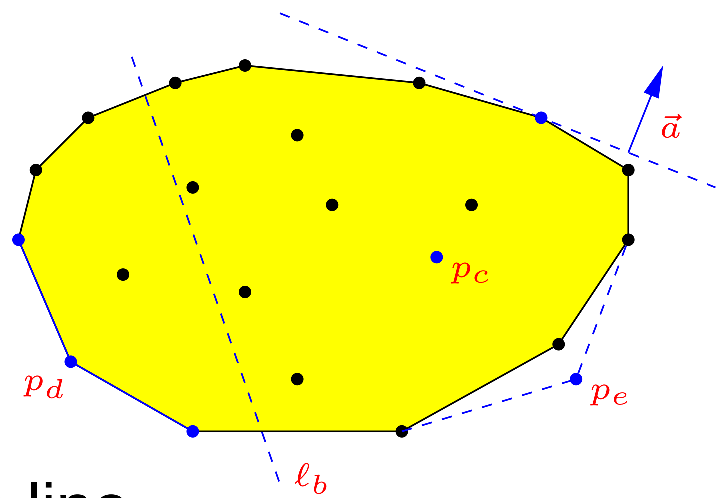
(a) The extreme point in a direction

(b) Does a line intersect $\mathrm{CH}(S)$?

(c) Is a point inside $\mathrm{CH}(S)$?

(d) Neighbor points on $\mathrm{CH}(S)$

(e) Tangent points on $\mathrm{CH}(S)$



(f) The edges of $\mathrm{CH}(S)$ intersected by a line

Dynamic Planar Convex Hull Results

Insertions only	Update	Query
Preparata 1979	$O(\log n)$	$O(\log n)$

Deletions only		
Hershberger, Suri 1992	$O_{\mathrm{A}}(\log n)$	$O(\log n)$

Offline		
Hershberger, Suri 1996	$O_{\mathrm{A}}(\log n)$	$O(\log n)$

Fully dynamic		
Overmars, van Leeuwen 1981	$O(\log^2 n)$	$O(\log n)$
Chan 1999	$O_{\mathrm{A}}(\log^{1+\epsilon} n)$	$O(\log n)$
Brodal, Jacob 2000 Kaplan,Tarjan,Tsioutsiouliklis '01	$O_{\mathrm{A}}(\log n \cdot \log\log n)$	$O(\log n)$

O_{A}=Amortized Query=Queries (a)–(e)

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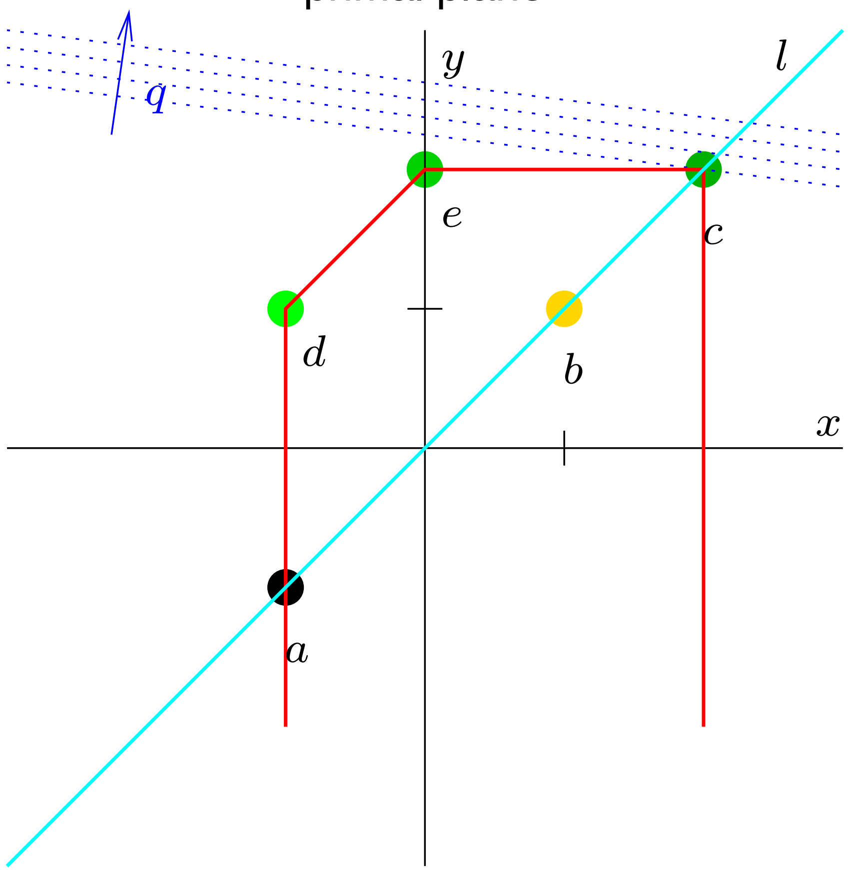
this Thesis	$O_{\mathrm{A}}(\log n)$	$O(\log n)$

O_{A}=Amortized   Query=Queries (a)–(e)

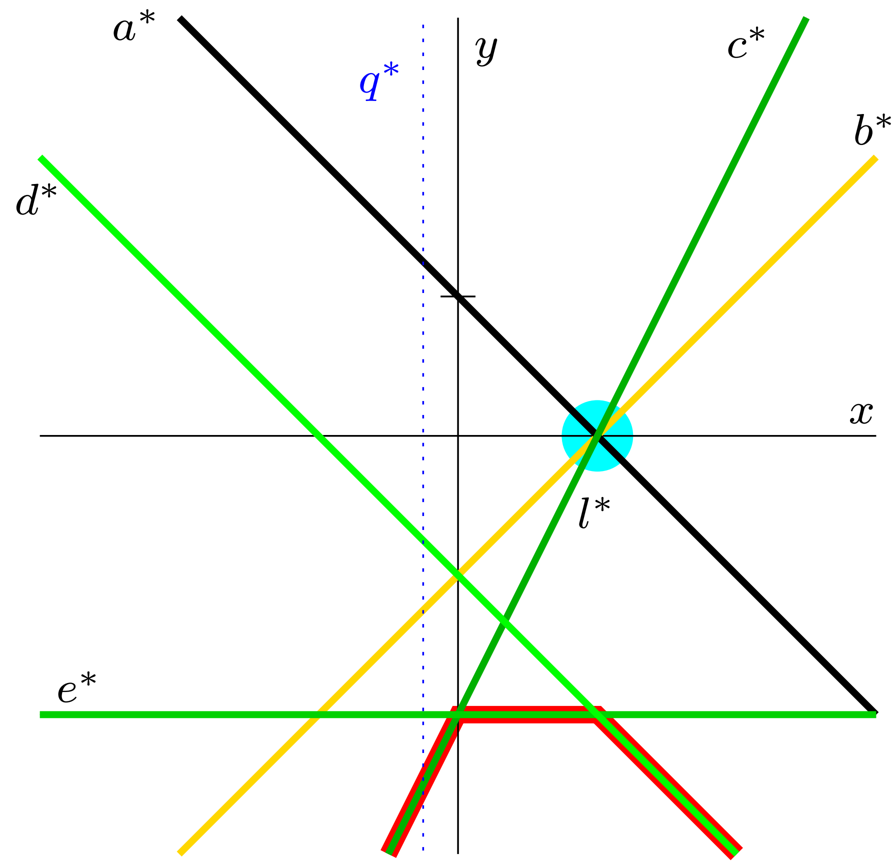
Duality Transformation

$$p = (a, b) \in \mathbb{R}^2 \quad \text{maps to} \quad p^* \;:=\; (a \cdot x - b = y)$$

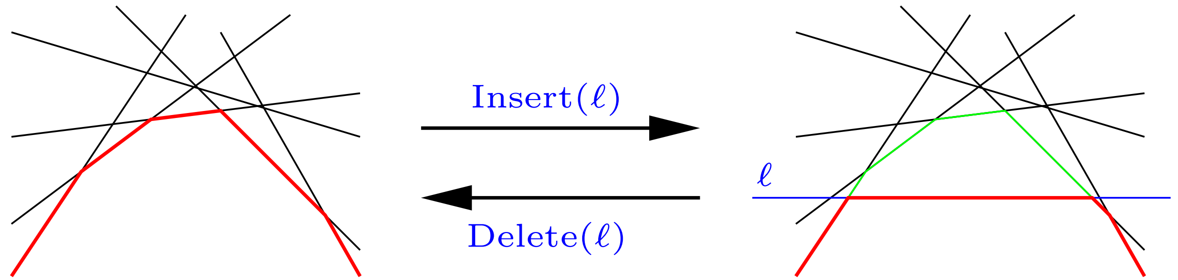
primal plane



dual plane

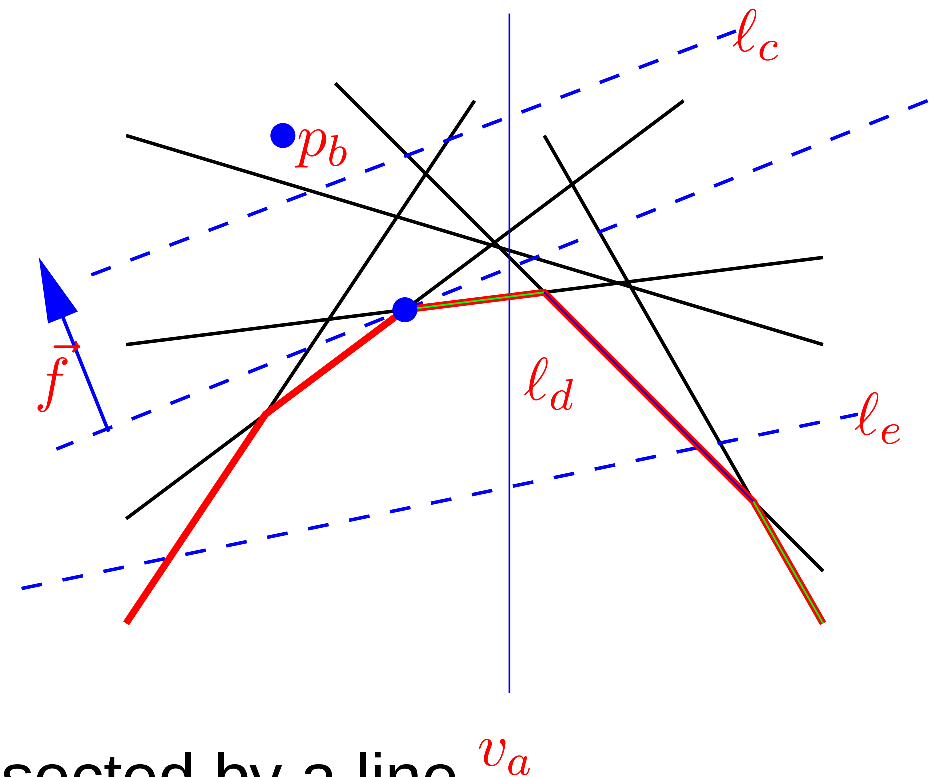


Dynamic Lower Envelope



Updates

Insert and delete lines



Queries

(a) Vertical line intersection

(b) Is a point above $\mathrm{LE}(S)$

(c) Is a line above $\mathrm{LE}(S)$

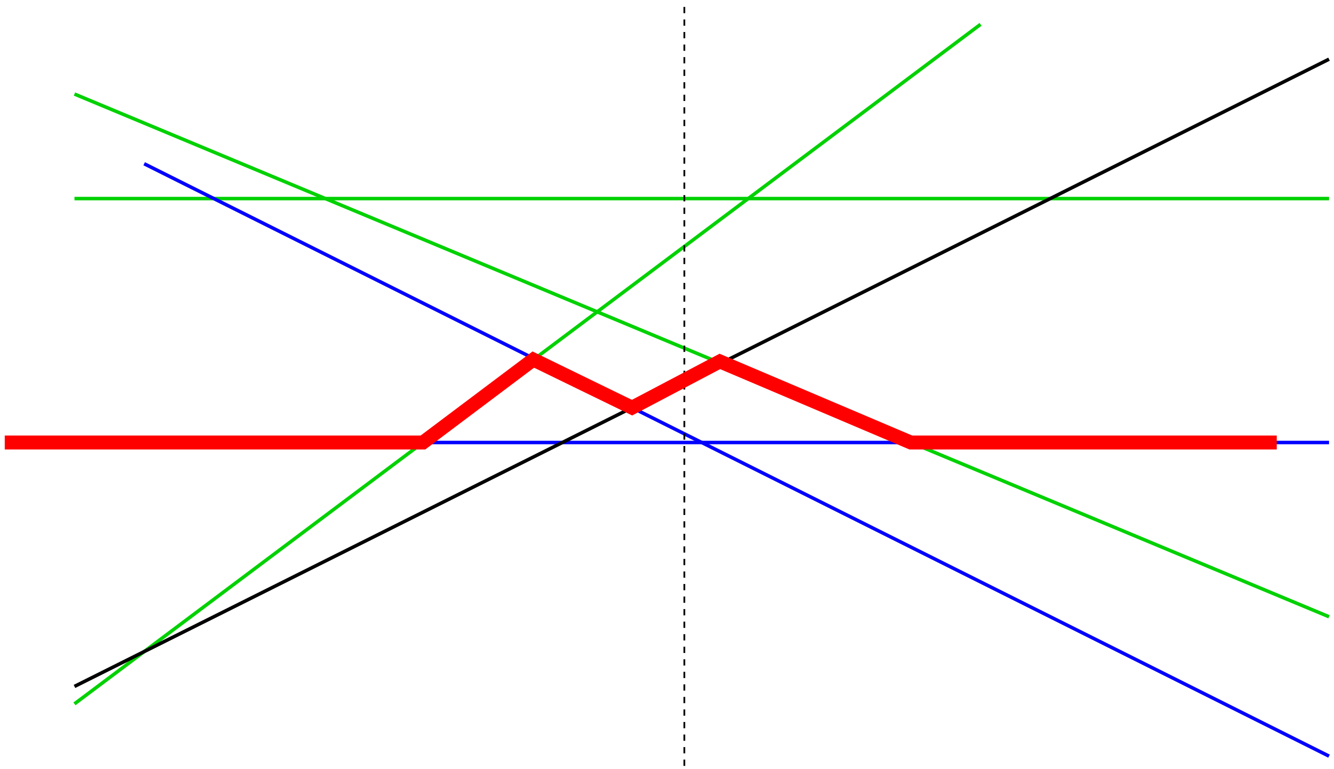
(d) Next segments on $\mathrm{LE}(S)$

(e) The segments of $\mathrm{LE}(S)$ intersected by a line

(f) The extreme point of $\mathrm{LE}(S)$ in some direction

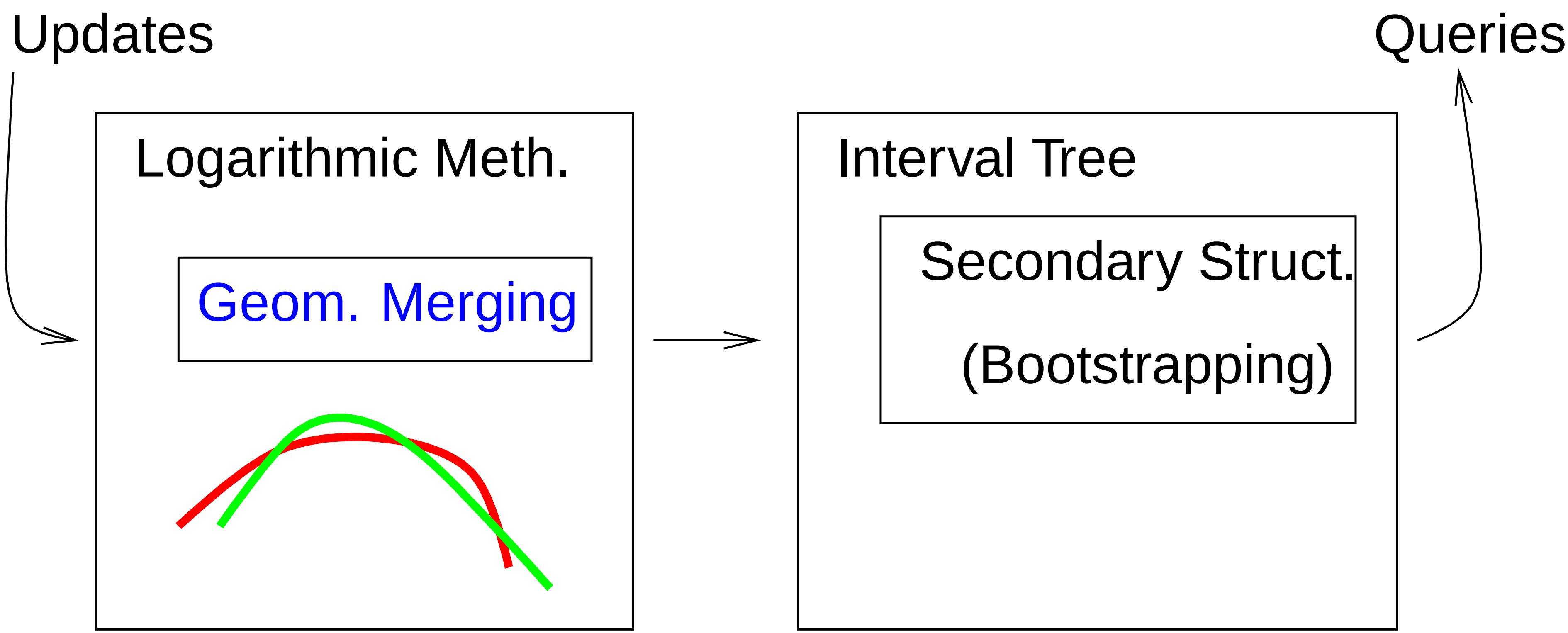
Application: k-level in the plane

Sweep-line algorithm Edelsbrunner and Welzl 1986
Previously $O((n+m)\alpha(n)\log n)$ expected time Har-Peled 1998
Now $O((n+m)\log n)$ for m segments on the k-level



The 3-level of the 6 lines is depicted in thick red.

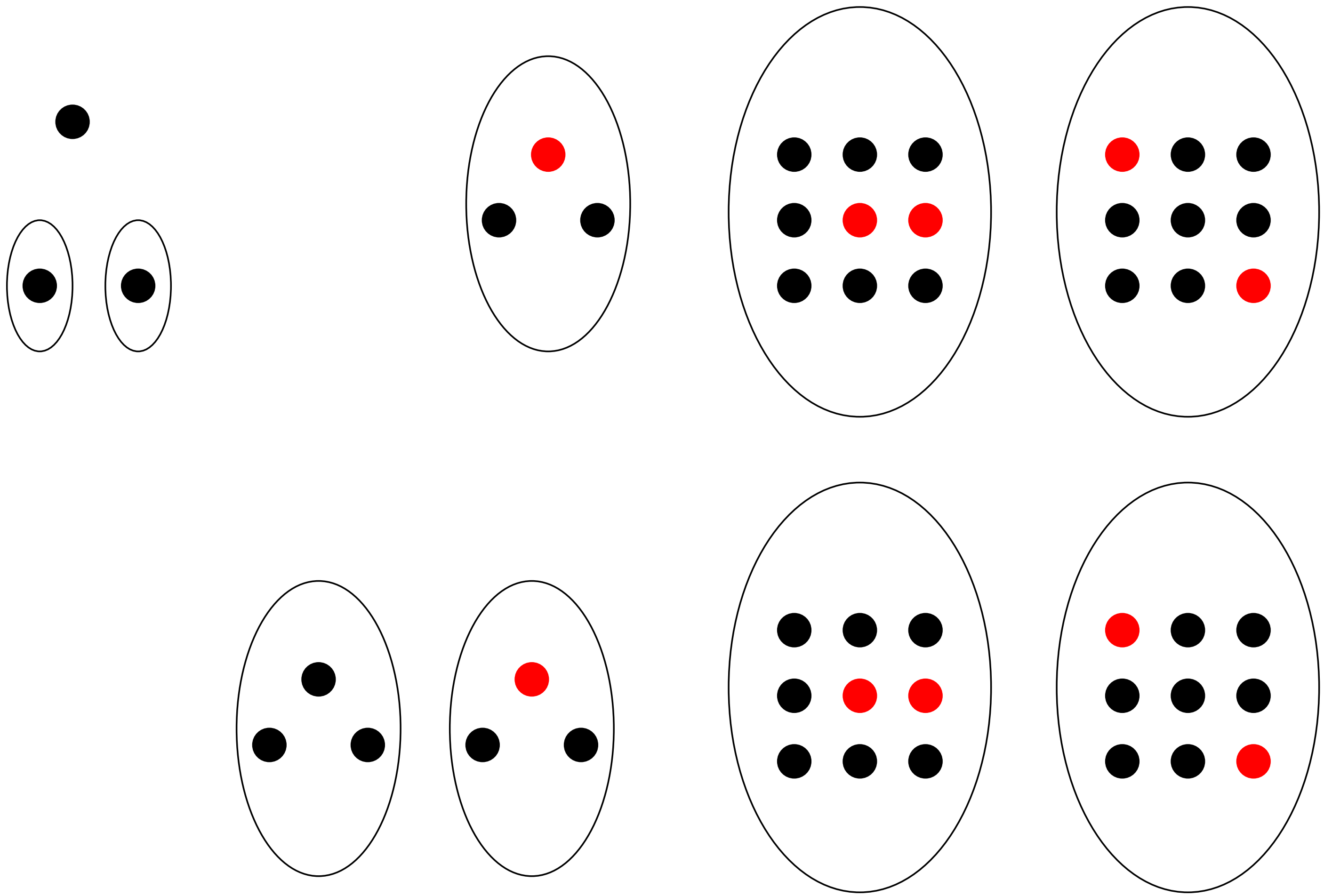
Overall Structure



$\log^2 n$ explicit hulls,
deletion-only

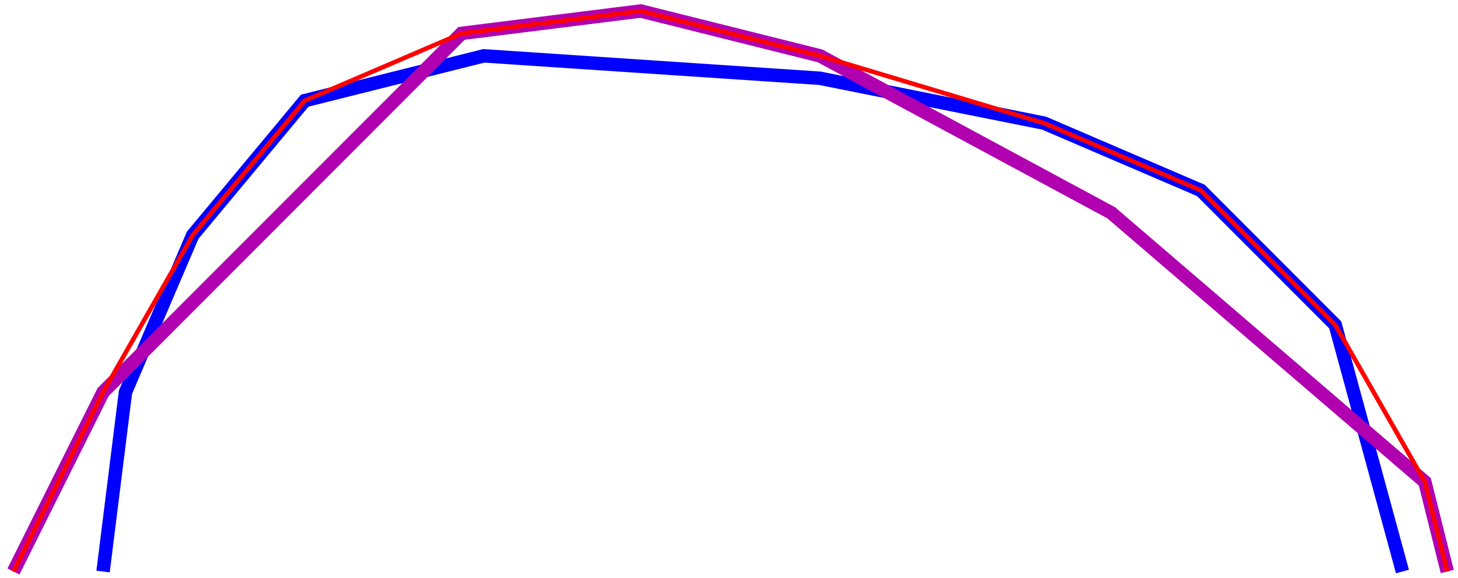
$2\times$ Bootstrapping:
Insert and Query $\log n$
Delete: $n \to \log^5 n \to \log n$

Logarithmic Method



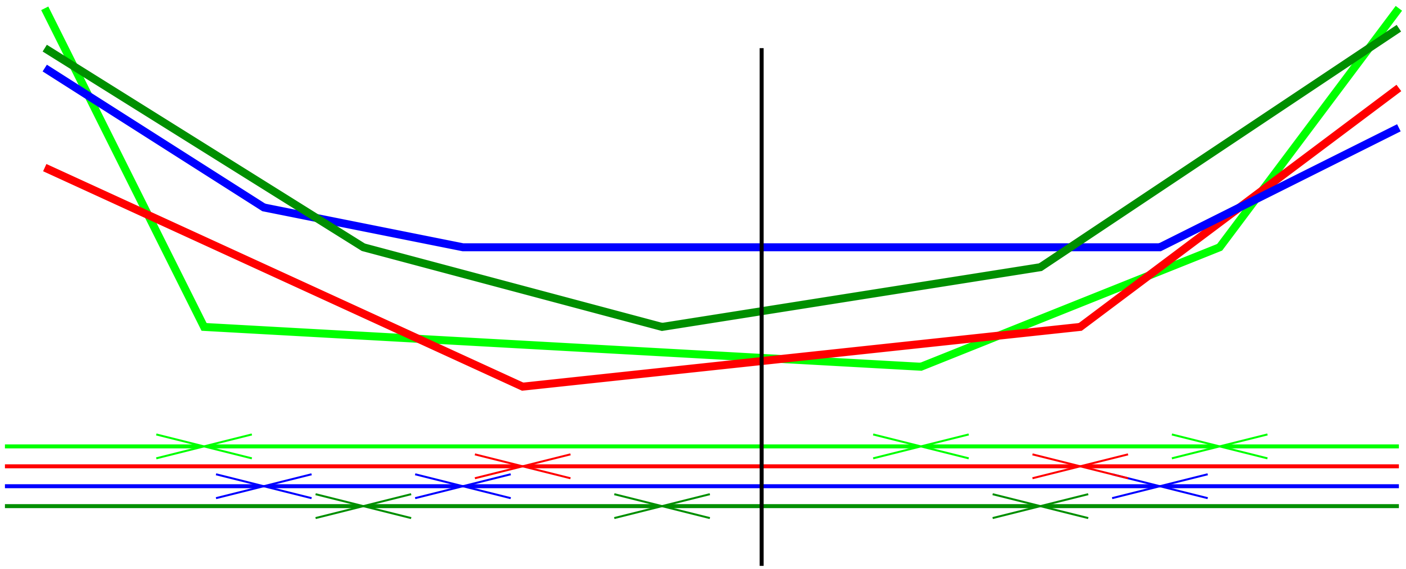
Bentley and Saxe 1980

Static Geometric Merging



Difficult: maintain under deletions of points

Combining Queries: Interval Tree



Task: combine the search on several lower envelopes into one search.
Follows ideas from Chan 1999; different choice of parameters, save some work by relaxed placement and lazy movements:

exploit knowledge about the (dynamic of the) intervals

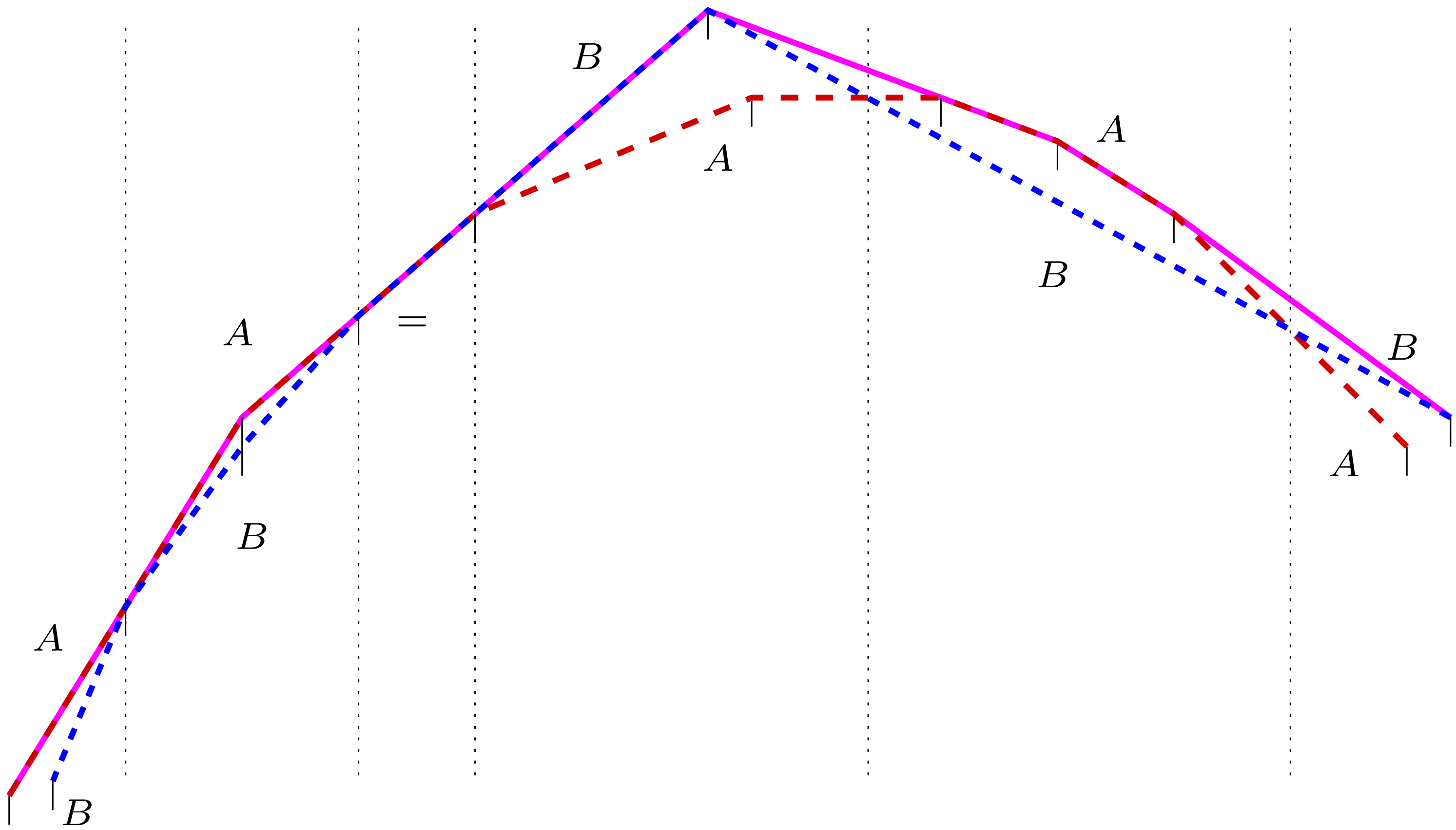
Semidynamic Merging

Create Set(p)	Create singleton set
Merge(A, B)	Combine data structures for A and B into one new for $A \cup B$
Delete(r)	Delete r from all merging structures

- Maintains list of points on the upper hull
- Works on binary merging forest
- Performance: $O_{\mathrm{A}}(1)$ per element in the set

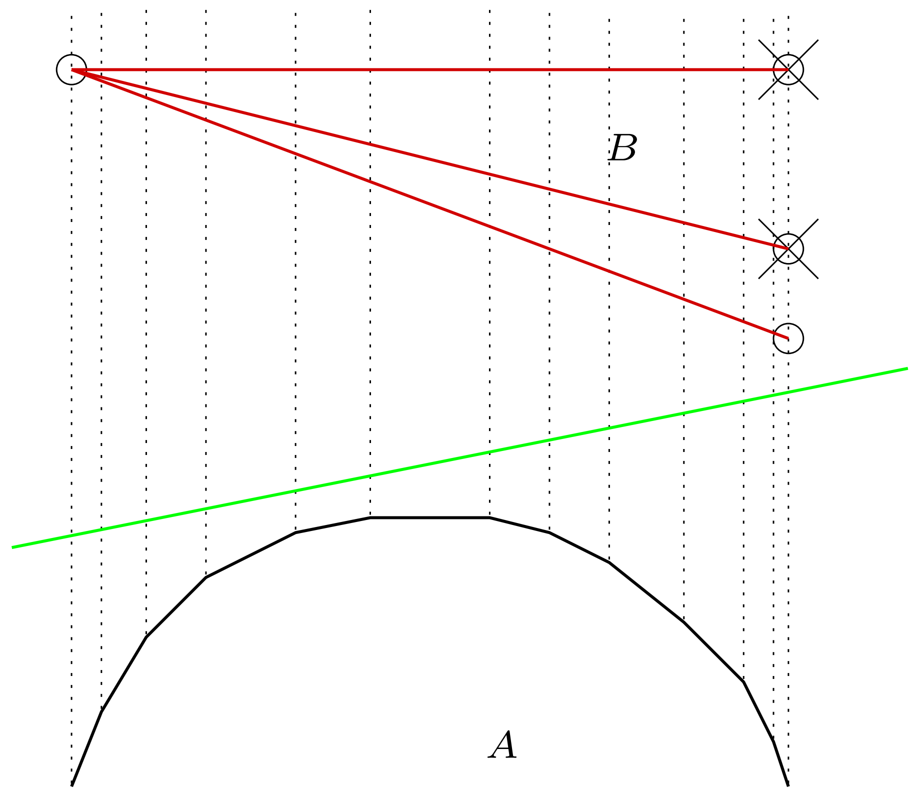
Core Problem

Maintain the equality points of the merging



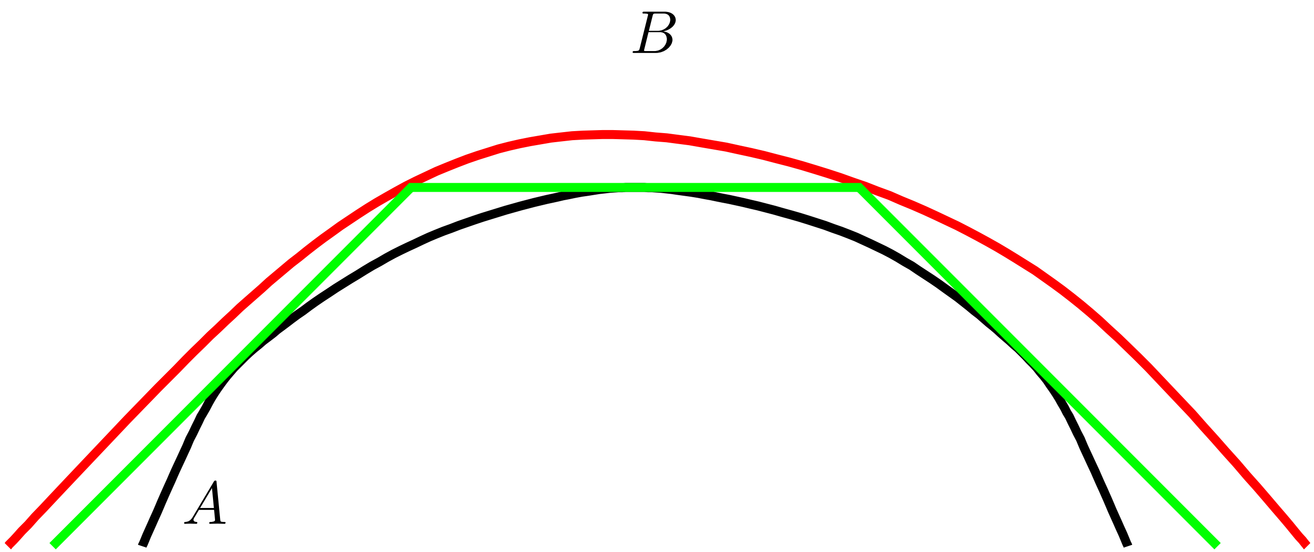
An equality oracle allows $O_{\mathrm{A}}(1)$ per element hull maintenance

Separation Certificate



vertical certificates (sweep line): $O(n)$ per deletion
parallel tangent search: $O(\log n)$ per deletion
suspended search: in some variant $O(1)$ per deletion

Greedy Separation



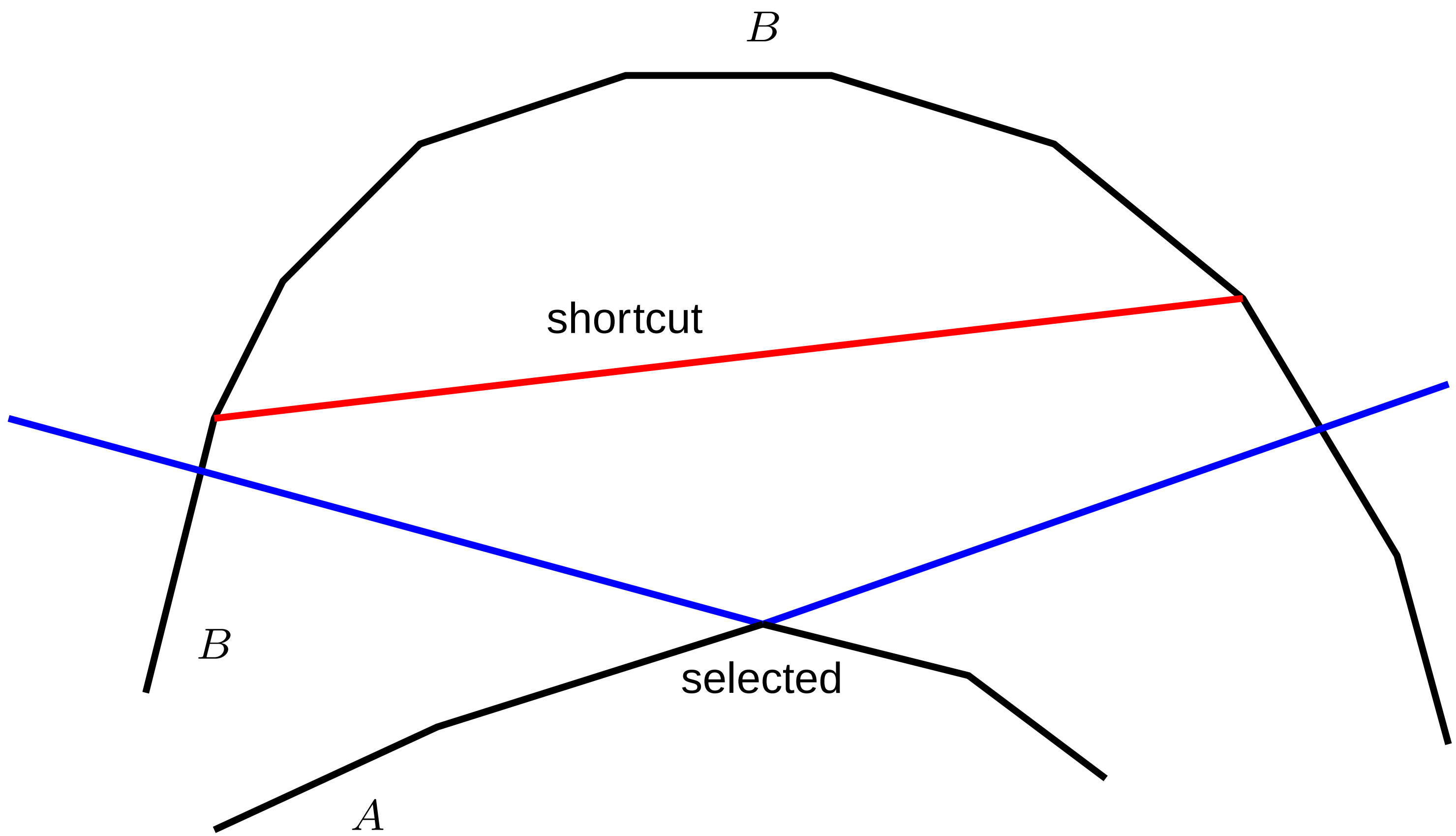
Greedily choosing tangent lines

Seems too rigid and sensitive for changes

Truss Bridge

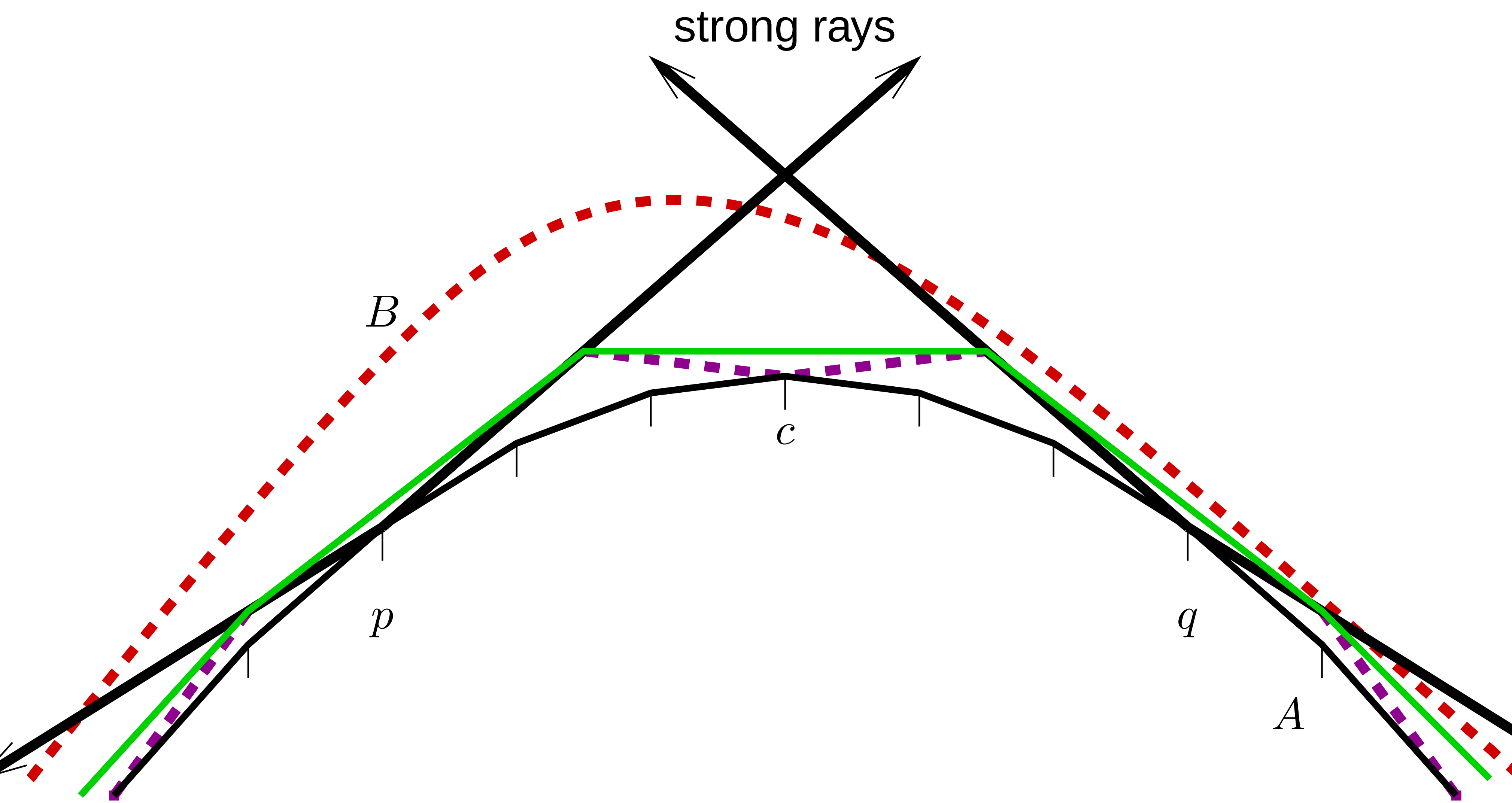
We call the construction Truss

Shortcuts



Shortcuts: Reducing the complexity of the outer hull

Dangling search



One deletion affects only constantly many strong rays

Splitters

Data structure(s) keeping (family of) sorted sequences

Elements e_i from a totally ordered universe

Build($e_1, \ldots, e_n$)	$O_{\mathrm{A}}(n)$
Split(t)	$O_{\mathrm{A}}(1)$
Extend(e_{n+1})	$O_{\mathrm{A}}(1)$

Hoffmann, Mehlhorn, Rosenstiehl, Tarjan 1986

Split includes searching;

Dangling searches are suspended searches;

promise to split when finishing the search.

Splitters

Data structure(s) keeping (family of) sorted sequences

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Build($e_1, \ldots, e_n$)		$O_{\mathrm{A}}(n)$
Split(t)	suspended	$O_{\mathrm{A}}(1)$
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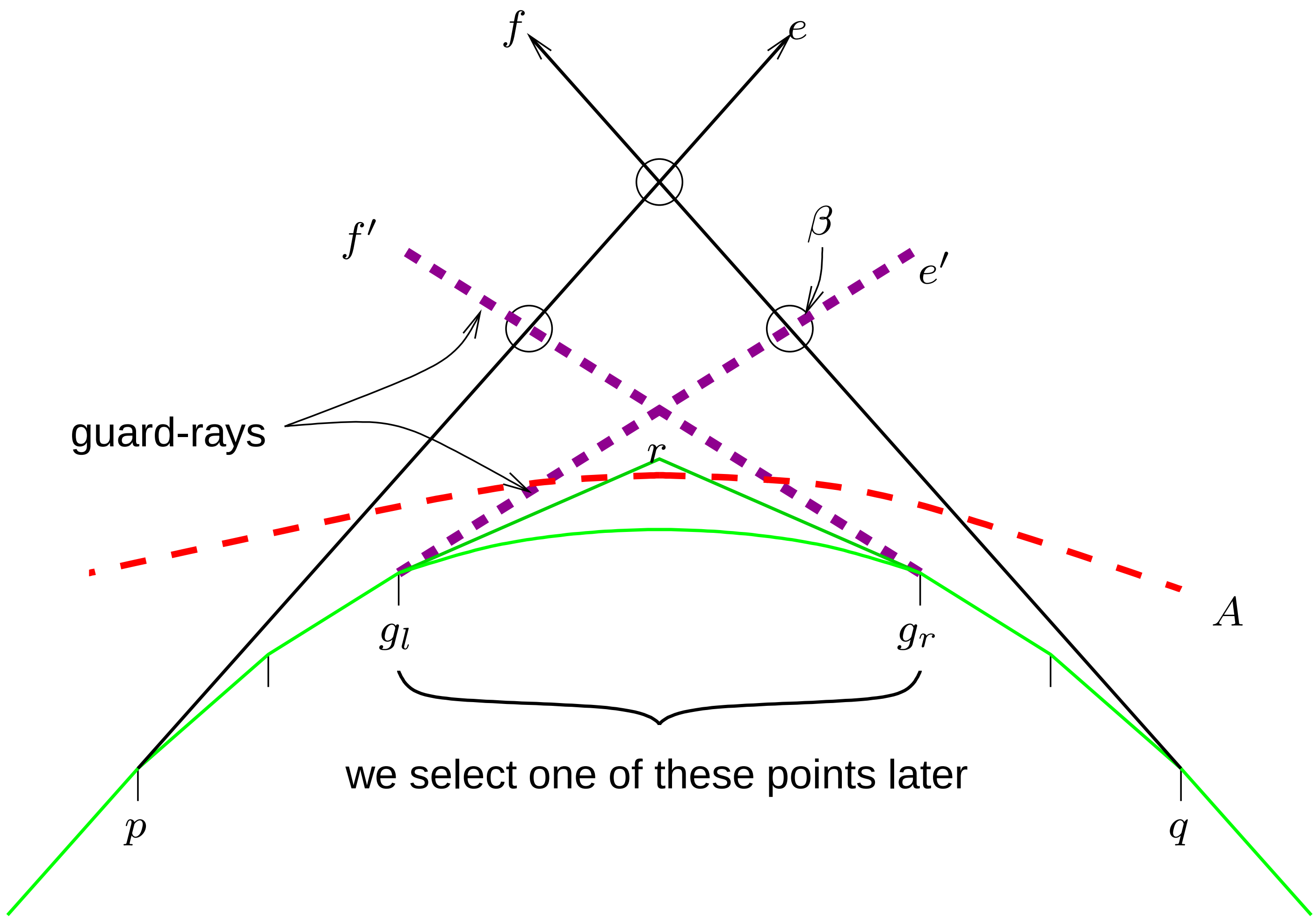
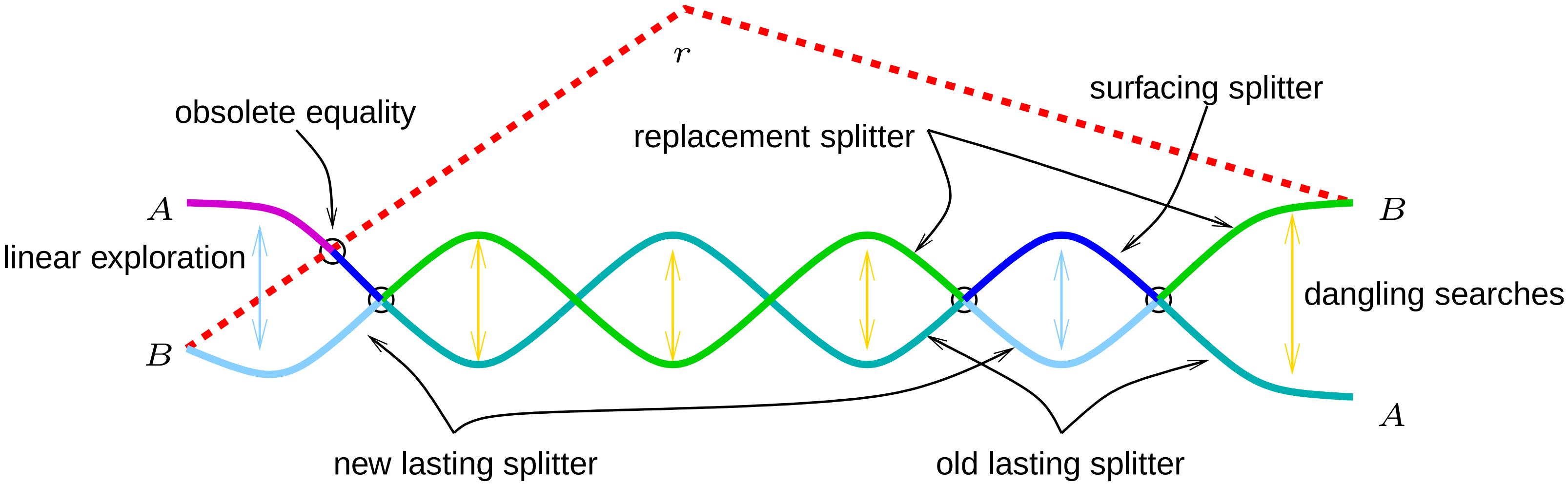
Joining Strips



The geometric situation of loosing equality points.
We join the splitter over a dangling search:
Feasible because we promise to split.

Account: Life-cycle of a point $p \in A$



Schematic: the two hulls after one deletion

1. p becomes part of $\mathrm{UH}(A)$.
   p in replacement splitter.
2. we realize $p \in \mathrm{UC}_0(B)$.
   p in lasting splitter.
3. we decide to select p.
4. Delete on B: $p \notin \mathrm{UC}(B)$.
   p in surfacing splitter.
5. p is hidden by a shortcut.
6. p is hidden by a bridge.
7. p gets on $UH(C)$.
8. The point p gets deleted.

New Techniques

Splitter: suspended (dangling) searches, restricted join over search

Dynamization: Reuse of existing data structures

Geometric Merging: Focus on equality points, selected points, dangling search, over-approximation, shortcuts, truss

Interval Tree: Relaxed placement of intervals, lazy movement, location justifier

Linear Space: Separators

Tight Lower Bounds

$q(n)$ be (amortized) query time

$I(n)$ amortized insertion time

$$q(n) = \Omega(\log n) \qquad \text{and} \qquad I(n) = \Omega\Big(\log \frac{n}{q(n)}\Big)\,.$$

on algebraic real-RAM, off-line usage of data structure, reduction based.

Applies to Membership and Predecessor as well.

Decision Problem

For $k < n$

$$(x_1, \ldots, x_n, y_1, \ldots, y_k) \in \text{DISJOINTSET}_{n,k} \subset \mathbb{R}^{n+k}$$

$$\iff$$

for all i, j we have $x_i \neq y_j$

$\text{DISJOINTSET}_{n,k}$has $\Omega(k^n)$ connected components

An algebraic computation tree has height $\Omega(n \log k)$.

(Ben-Or 83)

Summary and Open Problems

Presented

Dynamic Planar Convex Hull Data structure

Query	$O(\log n)$
Insert	$O_{\mathrm{A}}(\log n)$
Delete	$O_{\mathrm{A}}(\log n)$

and a matching lower bound.

Open Problems

- Make it simple
- worst-case instead of amortized bounds
- more general queries
- explicit maintenance of the hull