Combinatorial algorithms for graphs and partially ordered sets

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Outline

1 Introduction
   - Outline of the thesis
   - Poset dimension
   - Vertex-edge-face posets and vertex-face posets

2 The order dimension of planar maps
   - Brightwell and Trotter’s results
   - The dimension of V-E-F posets
   - The dimension of vertex-face posets
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2. The order dimension of planar maps
   - Brightwell and Trotter’s results
   - The dimension of V-E-F posets
   - The dimension of vertex-face posets
The dissertation consists of four parts:

1. Reachability oracles
2. Reachability substitutes
3. The order dimension of planar maps
4. Approximation algorithms for graphs with large treewidth
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Partially ordered sets

A partially ordered set (poset) is a pair $P = (X, P)$ of a ground set $X$ (the elements of the poset) and a binary relation $P$ on $X$ that is

- transitive ($a \leq b$ and $b \leq c$ implies $a \leq c$),
- reflexive ($a \leq a$) and
- antisymmetric ($a \leq b$ implies $b \not\leq a$ ($a \neq b$))
Posets are often represented by their diagrams.

Example

\[ c \leq a, \]
\[ d \leq a, \]
\[ e \leq d, \]
\[ d \leq b \]
Let $\mathbf{P} = (P, X)$ be a poset.

**Definition**

A linear extension $L$ of $P$ is a linear order that is an extension of $P$, i.e., $x \leq_P y \Rightarrow x \leq_L y$. 
Linear extensions

Example

\[ a \quad b \quad c \quad d \quad e \]

\[ a \quad c \quad b \quad d \quad e \]
### Definition

A family of linear extensions $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of $P$ is a realizer of $P$ if $P = \cap \mathcal{R}$. The dimension of $P$ is the minimum cardinality of a realizer of $P$. 
Dimension

Example

The order dimension of planar maps

Summary

Outline of the thesis

Poset dimension

Vertex-edge-face posets and vertex-face posets
Why is dimension interesting?

- Measures how close a poset is to being a linear order.
- Low dimension implies a compact representation.

Example

<table>
<thead>
<tr>
<th>$a$</th>
<th>$(5, 4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b$</td>
<td>$(3, 5)$</td>
</tr>
<tr>
<td>$c$</td>
<td>$(4, 1)$</td>
</tr>
<tr>
<td>$d$</td>
<td>$(2, 3)$</td>
</tr>
<tr>
<td>$e$</td>
<td>$(1, 2)$</td>
</tr>
</tbody>
</table>
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Planar maps

A planar map is the sets of vertices (points), edges (lines) and faces (regions) of a crossing-free drawing of a graph in the plane and the incidences between those sets.

The dual map $M^*$ of a planar map $M$ is a planar map with a vertex for each face in $M$ and a face for each vertex in $M$ like in this example.
Planar maps

Example

$M$

$M^*$

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Combinatorial algorithms for graphs and partially ordered sets
Outerplanar maps

If all the vertices are on the outer face, the map is strongly outerplanar.

If there is a different drawing of the same graph where all the vertices are on the outer face, the map is weakly outerplanar.

Example
Vertex-edge-face and vertex-face posets

Definition

The vertex-edge-face poset $P_M$ of a planar map $M$ is the poset on the vertices, edges and faces of $M$ ordered by inclusion.

The vertex-face poset $Q_M$ of $M$ is the subposet of $P_M$ induced by the vertices and faces of $M$. 
Vertex-edge-face and vertex-face posets

Example

\[ M \]

\[ F_{\Delta} \]

\[ F_{\infty} \]

\[ Q_{M} \]

\[ P_{M} \]

\[ F_{\Delta} \]

\[ F_{\infty} \]
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The order dimension of planar maps

Summary

Brightwell and Trotter’s results
The dimension of V-E-F posets
The dimension of vertex-face posets

The Brightwell-Trotter Theorems

Theorem (Brightwell & Trotter)

Let $M$ be a planar map. Then $\dim(P_M) \leq 4$.

Theorem (Brightwell & Trotter)

Let $M$ be a 3-connected planar map. Then $\dim(Q_M) = 4$. 
The Brightwell-Trotter Theorems

Theorem (Brightwell & Trotter)

Let $M$ be a planar map. Then $\dim(P_M) \leq 4$.

Theorem (Brightwell & Trotter)

Let $M$ be a 3-connected planar map. Then $\dim(Q_M) = 4$. 
Two questions of Brightwell and Trotter

1. For which planar maps is \( \dim(P_M) \leq 3 \)?

2. For which planar maps is \( \dim(Q_M) \leq 3 \)?

We know when the dimension is at most 2.
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3-dimensional V-E-F posets of planar maps

Theorem (Felsner & N.)

Let $M$ be a planar map such that $\dim(P_M) \leq 3$. Then both $M$ and the dual map $M^*$ are outerplanar.
Observation: If \( M \) is connected, \( P_{M^*} = (P_M)^* \).
3-dimensional V-E-F posets of planar maps

Proof (sketch).

A map is outerplanar if it does not contain a $K_4$-subdivision or $K_{2,3}$-subdivision.
3-dimensional V-E-F posets of planar maps

Proof (sketch).

If $M$ contains a subdivision of $K_4$, then the vertex-face poset of some 3-connected map is a subposet of $Q_M$. Use the second Brightwell-Trotter Theorem.
3-dimensional V-E-F posets of planar maps

Proof (sketch).
Suppose $M$ contains a subdivision of $K_{2,3}$.

\[
P_1 \quad P_2 \quad P_3
\]

$u \quad \quad v$
3-dimensional V-E-F posets of planar maps

Proof (sketch).

The three paths $P_1$, $P_2$ and $P_3$ induces three mutually disjoint fences in $P_M$. 

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{proof_sketch.png}
\end{figure}

$u$ $v$
Critical pairs

Definition

A critical pair is a pair of incomparable elements \((a, b)\) such that \(x < b\) if \(x < a\) and \(y > a\) if \(y > b\) for all \(x, y \in X \setminus \{a, b\}\).

Example

![Diagram of critical pairs](image)
Fact

A family of linear extensions $\mathcal{R} = \{ L_1, L_2, \ldots, L_t \}$ of $P$ is a realizer of $P$ iff for each critical pair $(a, b)$ there is some $L \in \mathcal{R}$ such that $b <_L a$. We then say that $(a, b)$ is reversed in $L$.

Example

![Diagram of a simple poset and its realization](image.png)
3-dimensional V-E-F posets of planar maps

Proof (sketch).

We then show that if $\dim(P_M) \leq 3$, then all the critical pairs of the poset below must reversed in a single linear extension.

But this poset has dimension 2.
Path-like maps

Definition

A 2-connected strongly outerplanar map with a weakly outerplanar dual is called path-like.

Example

A 2-connected simple outerplanar map has a unique Hamilton cycle. We can partition the edges into cycle edges and chordal edges.
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Alternating cycles

Definition

An alternating cycle is a sequence of critical pairs \((a_0, b_0), \ldots, (a_k, b_k)\) such that \(a_i \leq b_{i+1} \mod (k+1)\) for all \(i = 0, \ldots, k\).

Example

\((b, a), (c, b)\) is an alternating cycle since \(b \leq b\) and \(c < a\).
Alternating cycles, dimension

Fact

Let $P$ be a poset. Then $\dim(P) \leq t$ iff there exists a $t$-coloring of the critical pairs of $P$ such that no alternating cycle is monochromatic.
Path-like maps

We can encode any 3-realizer of the V-E-F poset of a maximal path-like map as an oriented 3-coloring of its chordal edges.

Example

However, not every oriented 3-coloring corresponds to a 3-realizer . . .
Path-like maps

Theorem (Felsner & N.)

Let $M$ be a maximal path-like map. Then $\dim(P_M) \leq 3$ if and only if the chordal edges of $M$ has a permissible coloring.
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An example of an outerplanar map with $\dim(Q_M) = 4$

- Vertex-face posets of dimension 3 are more complicated.
- We still cannot have a subdivision of $K_4$ contained in the map.
- Even showing the existence of a strongly outerplanar map with $\dim(Q_M) = 4$ is a bit of work.
An example of an outerplanar map with \( \text{dim}(Q_M) = 4 \)

**Theorem (Felsner & N.)**

*There is an outerplanar map \( M \) with \( \text{dim}(Q_M) = 4 \).*
An example of an outerplanar map with $\dim(Q_M) = 4$

- 3-color the critical pairs of type (vertex, bounded face).
- All vertices are on the outer face, so the critical pairs of a bounded face cannot have all 3 colors.
- All 3 colors must appear around a strongly interior face.
An example of an outerplanar map with $\text{dim}(Q_M) = 4$
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An example of an outerplanar map with $\text{dim}(Q_M) = 4$
If \( \dim(\mathcal{P}_M) \leq 3 \), then \( M \) and \( M^* \) are outerplanar.

If \( M \) is a maximal path-like map, \( \dim(\mathcal{P}_M) \leq 3 \) iff \( M \) has a permissible coloring.

There are strongly outerplanar maps \( M \) with \( \dim(\mathcal{Q}_M) = 4 \).
If $\dim(P_M) \leq 3$, then $M$ and $M^*$ are outerplanar.

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There are strongly outerplanar maps $M$ with $\dim(Q_M) = 4$. 
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