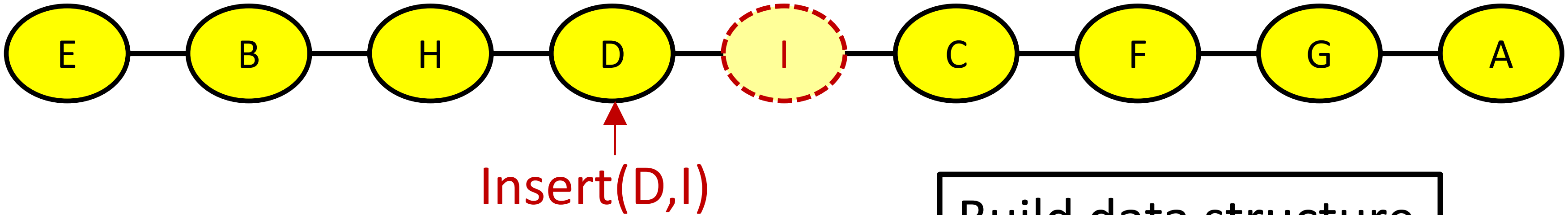
List Order Maintenance

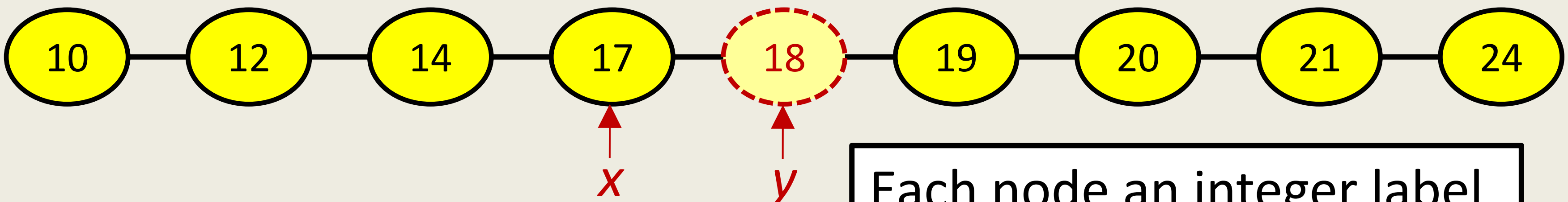


Build data structure

Insert(*x*,*y*) Insert *y* after *x*

Order(*x*,*y*) Returns if *x* is to the left of *y*

Monotonic List Labeling

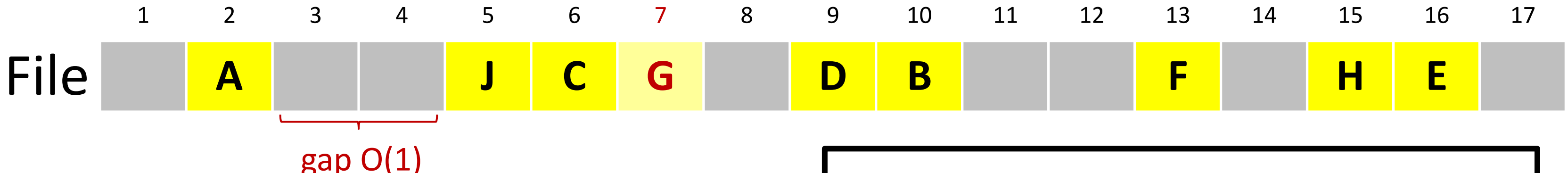


Each node an integer label

Relabel nodes on insertion

Insert(*x*,*y*) Insert *y* after *x*

Density Maintenance



Insert(*i*,*x*) Insert *x* at postion *i*

Shift elements on insertion

List Order Maintenance

[P. Dietz, D. Sleator, *Two algorithms for maintaining order in a list*, ACM Conference on Theory of Computing, 365-372, 1987]

[A. Tsakalidis, *Maintaining Order in a Generalized Linked List*. Acta Informatica 21: 101-112, 1984]

Query and Insert O(1)

Monotonic List Labeling

[P. Dietz, *Maintaining Order in a Linked List*, ACM Conference on Theory of Computing, 122-127, 1982]

[P. Dietz, J. Seiferas, J. Zhang: *A Tight Lower Bound for On-line Monotonic List Labeling*. Scandinavian Workshop on Algorithm Theory, 131-142, 1994]

Max label $O(n^k)$, $k>1+\varepsilon$ $\Theta(\log n)$ relabelings

[D. Willard, *Maintaining Dense Sequential Files in a Dynamic Environment*, ACM Conference on Theory of Computing, 114-121, 1982]

[P. Dietz, J. Zhang: *Lower Bounds for Monotonic List Labeling*. Scandinavian Workshop on Algorithm Theory, 173-180, 1990]

Max label $O(n)$ $\Theta(\log^2 n)$ relabelings

Applications

[G. Brodal, R. Fagerberg, R. Jacob, *Cache-Oblivious Search Trees via Binary Trees of Small Height,* ACM-SIAM Symposium on Discrete Algorithms, pages 39-48, 2002]

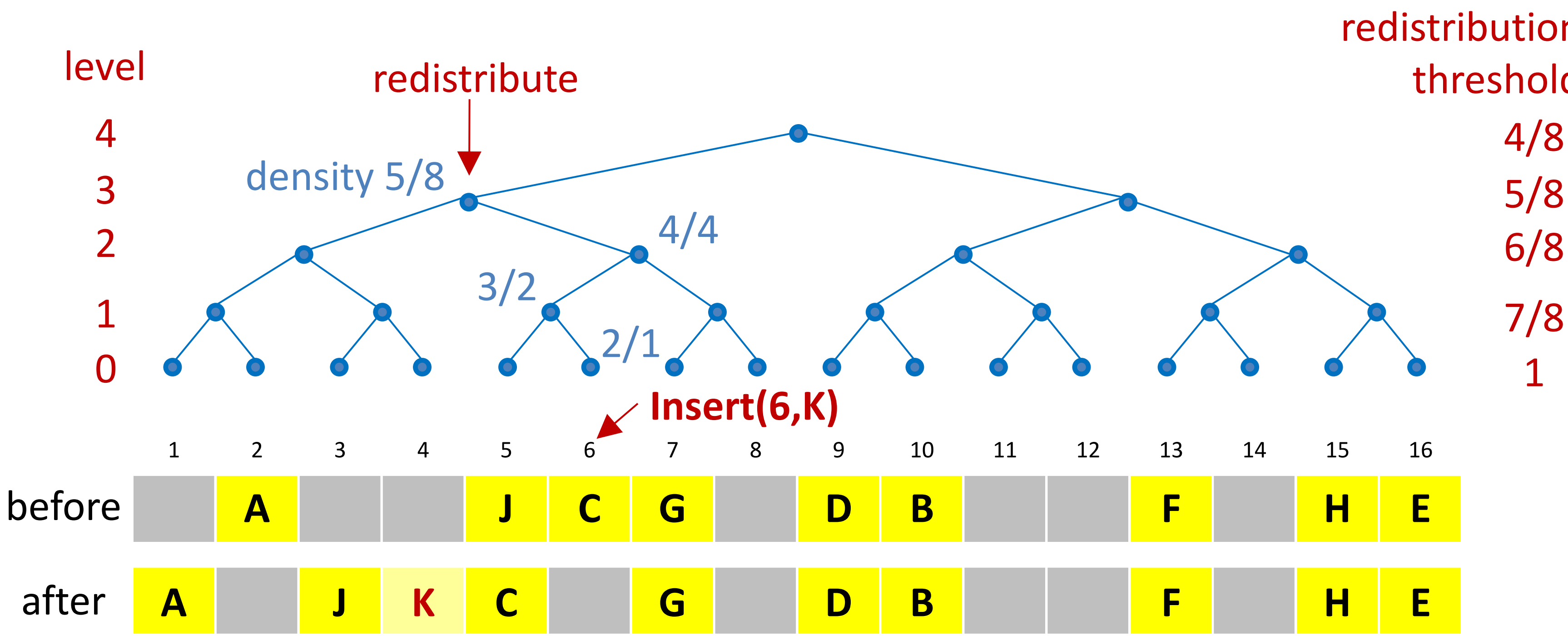
[J. Driscoll, N. Sarnak, D. Sleator, R. Tarjan, *Making Data Structures Persistent*, Journal of Computer and System Sciences, 38(1), 86-124, 1989]

Amortized O($\log^2 n$) Density Maintenance

- Threshold $\tau = 1/(2\log n)$
- Level i node overflows if density > $1-i\cdot\tau$
- **Insert** redistribute lowest non-overflowing ancestor

  $\Rightarrow$ a child requires τ fraction insertions before next overflow

  $\Rightarrow$ amortized insertion cost = #levels $\cdot$ 1 / τ = O($\log^2 n$)



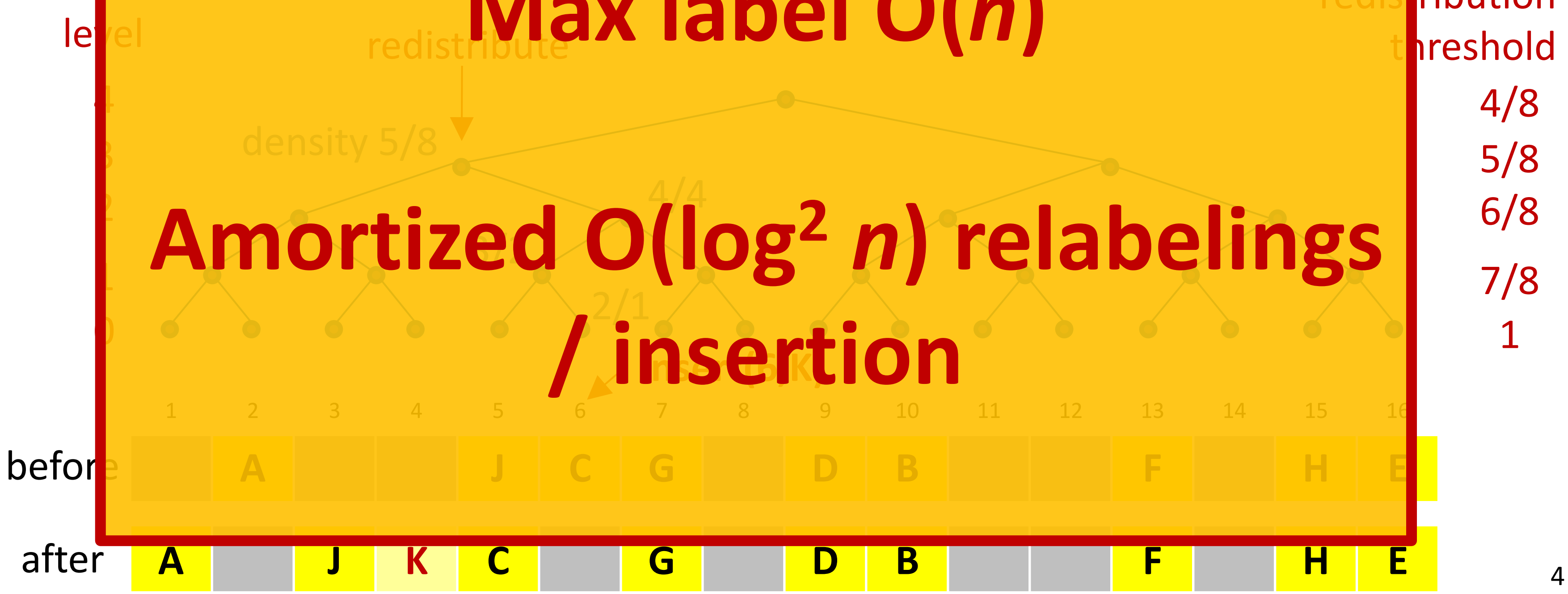
Amortized O($\log^2$ *n*) Density Maintenance

- Threshold $\tau = 1/(2\log n)$
- Level *i* node overflows if density $> 1 - i \cdot \tau$
- **Insert**: redistribute lowest non-overflowing ancestor
  - $\Rightarrow$ a child requires τ fraction insertions before next overflow
  - $\Rightarrow$ amoritzed insertion cost = #levels $\cdot$ 1 / τ = O($\log^2$ *n*)

$\Rightarrow$ List Order Maintenance

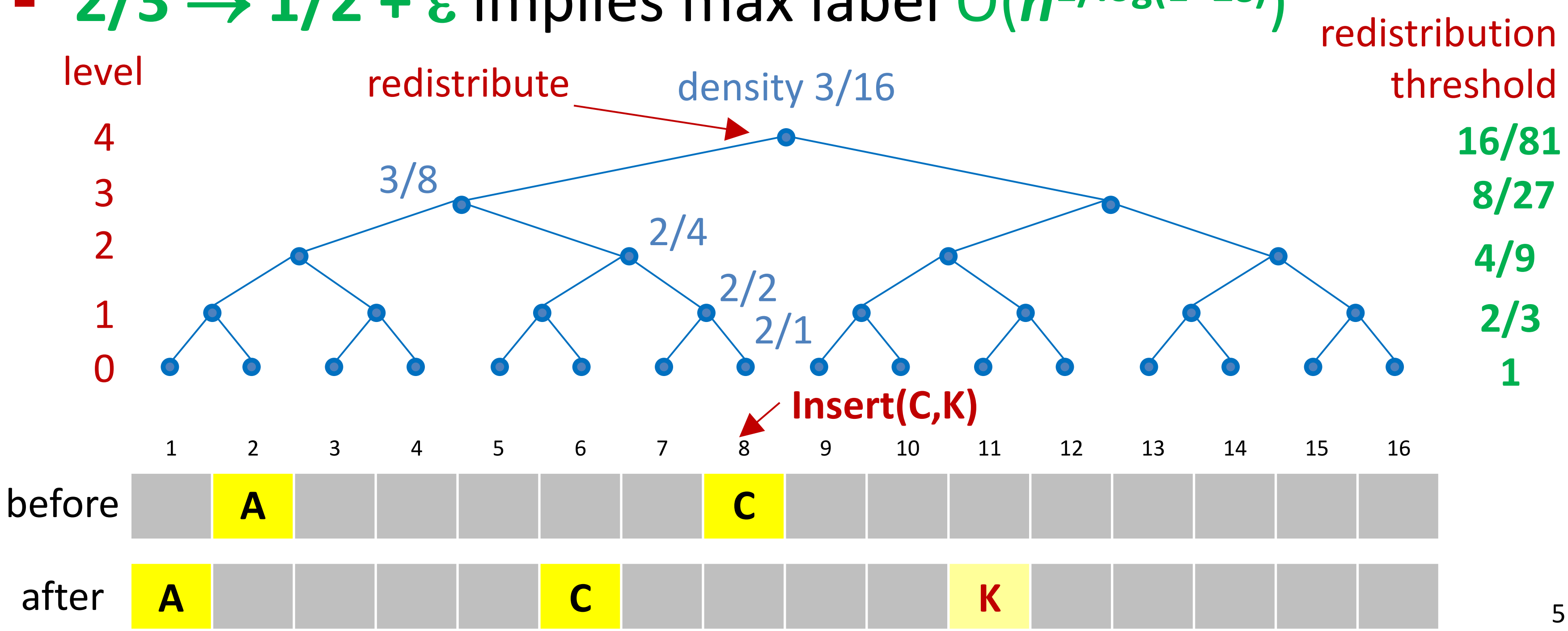
Max label O(*n*)

Amortized O($\log^2$ *n*) relabelings / insertion



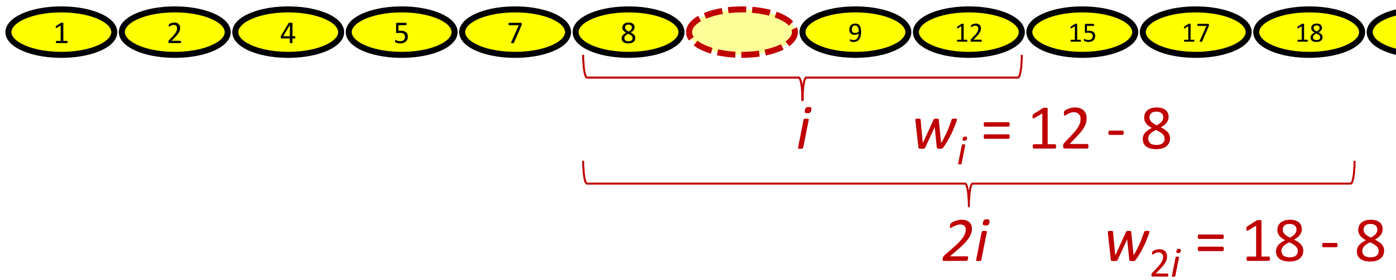
Amortized O(log n) List Relabelings

- Level i node overflows if density > $(2/3)^i$
- **Insert** redistribute lowest non-overflowing ancestor
  - ⇒ **≤ $\log_{4/3}$ n** levels ⇒ max label $2^{\log_{4/3} n} \leq n^{2.41}$
  - ⇒ a child requires **1/2** fraction insertions before next overflow
  - ⇒ amortized insertion cost = **#levels** · **3** = O(log n)
- **2/3 → 1/2 + ε** implies max label $O(n^{1/\log(1+2\varepsilon)})$



Amortized O(log *n*) List Relabelings

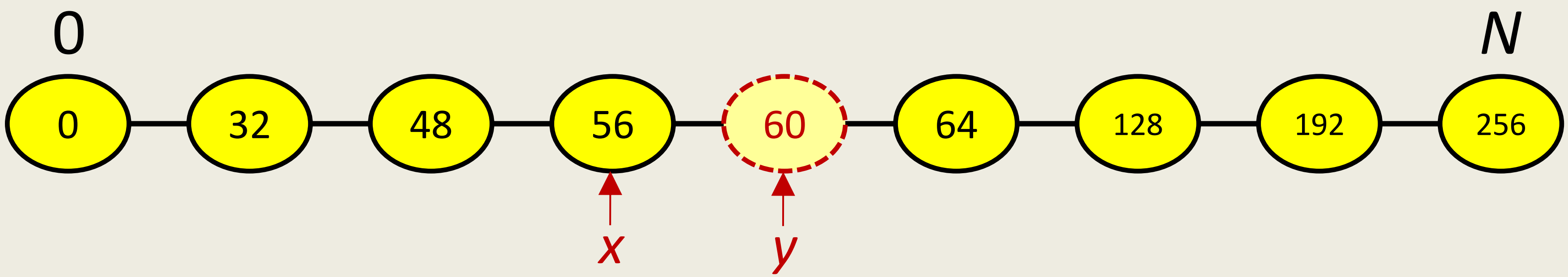
[P. Dietz, D. Sleator, *Two algorithms for maintaining order in a list*, ACM Conference on Theory of Computing, 365-372, 1987]



i = 1
while $w_{2i} \leq 4 \cdot w_i$ **do**
  $i = i + 1$
Relabel uniformly "2*i* area"

- Only relabels to the **right**
- Max label ***M*=4*n*²** ($M=4n^2$)
- Requires labels **mod *M*+1**

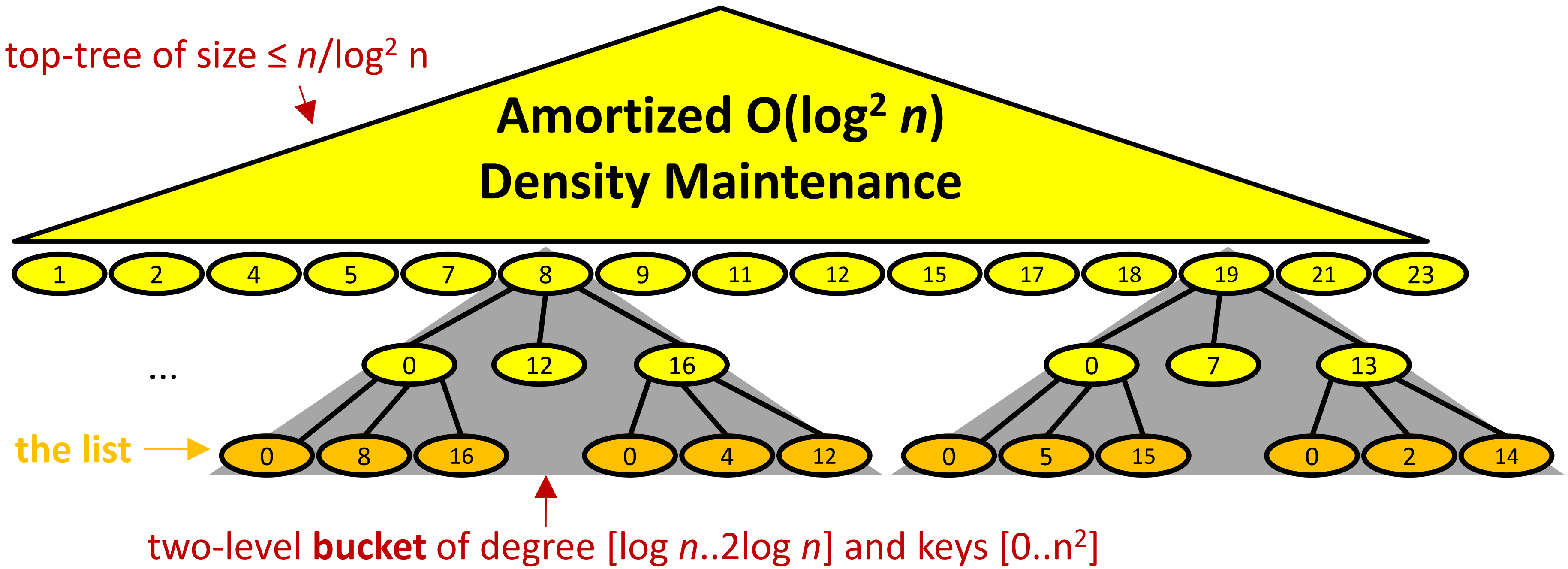
Monotonic List Labeling
O(log *N*) easy insertions



Insert(*x*,*y*) Label *y* = (left + right)/2

⇒ Can perform log *N* insertions without relabeling

Amortized O(1) List Order Maintenance



Insertion

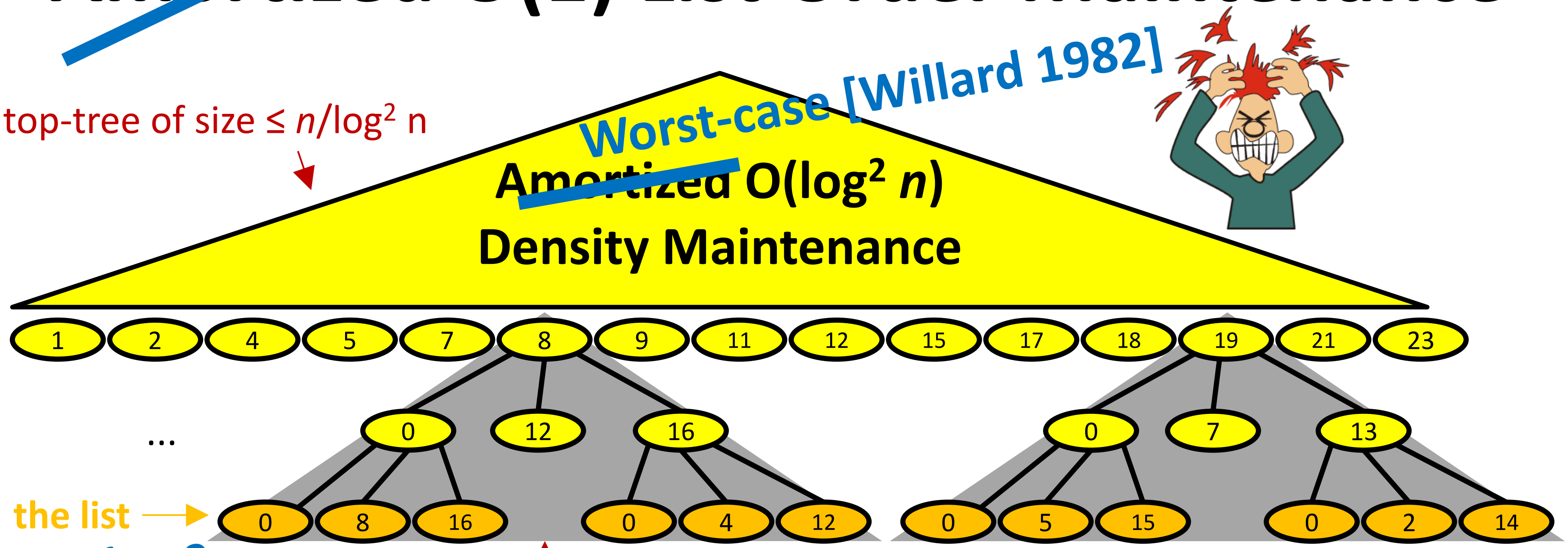
– create and label new **leaf**

– split nodes of degree > 2log n and relabel with gap n

– insert in top tree

~~Amortized~~ Worst-case O(1) List Order Maintenance

top-tree of size ≤ $n/\log^2 n$



the list →

~~two~~ three-level **bucket** of degree [log n..2log n] and keys [0..n^2]

Insertion

– create and label new leaf

– split nodes of degree > 2log n and relabel with gap n

– insert in top tree

+ incremental insertion into top tree
+ incremental splitting of bucket nodes
+ every O($\log^2 n$)'th operation split largest bucket

⇒ largest bucket size O($\log^3 n$)

Theorem 5 *Let $x_1, \ldots, x_n$ be n real valued variables, all initially zero. Repeatedly perform the following procedure:*

1. *Find an i, $1 \leq i \leq n$, such that $x_i = \max_j \{x_j\}$. Set x_i to zero.*
2. *Pick n nonnegative reals $a_1, \ldots, a_n$ such that $\sum_{i=1}^{n} a_i = 1$.*
3. *For $i = 1, \ldots, n$, set x_i to $x_i + a_i$.*

No x_i will ever exceed $H_{n-1} + 1$, where $H_k = \sum_{i=1}^{k} i^{-1}$, the kth harmonic number.