Priority Queues

- MakeQueue — create new empty queue
- Insert(Q,k,p) — insert key k with priority p
- Delete(Q,k) — delete key k (given a pointer)
- DeleteMin(Q) — delete key with min priority
- Meld(Q_1,Q_2) — merge two sets
- Empty(Q) — returns if empty
- Size(Q) — returns #keys
- FindMin(Q) — returns key with min priority

Priority Queues – Ideal Times

MakeQueue, Meld, Insert, Empty, Size, FindMin: $O(1)$

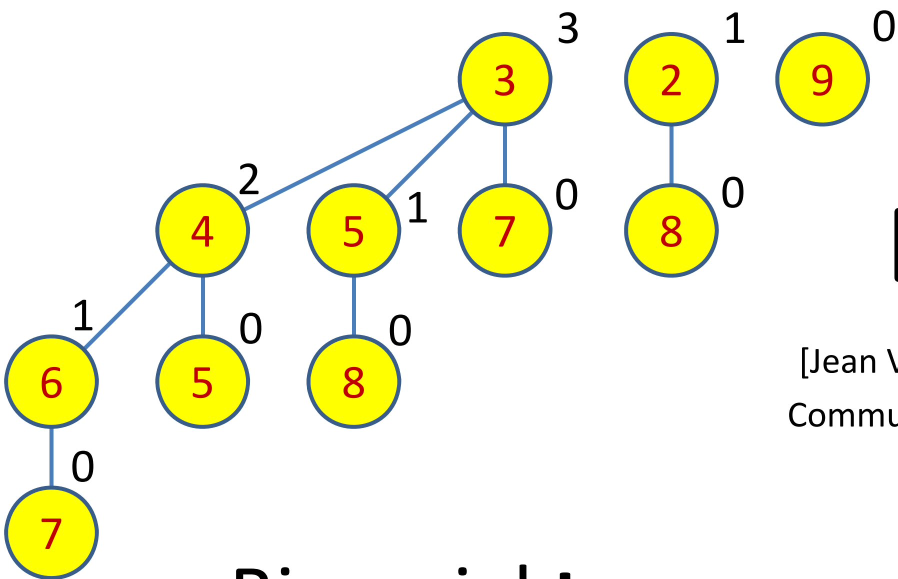
Delete , DeleteMin: $O(\log n)$

Thm

1) Meld $O(n^{1-\varepsilon}) \;\Rightarrow\;$ DeleteMin $\Omega(\log n)$

2) Insert, Delete $O(t) \;\Rightarrow\;$ FindMin $\Omega(n/2^{O(t)})$

1) Follows from $\Omega(n \cdot \log n)$ sorting lower bound
2) [G.S. Brodal, S. Chaudhuri, J. Radhakrishnan, *The Randomized Complexity of Maintaining the Minimum.* In Proc. 5th Scandinavian Workshop on Algorithm Theory, volume 1097 of Lecture Notes in Computer Science, pages 4-15. Springer Verlag, Berlin, 1996]



Binomial Queues

[Jean Vuillemin, *A data structure for manipulating priority queues,* Communications of the ACM archive, Volume 21(4), 309-315, 1978]

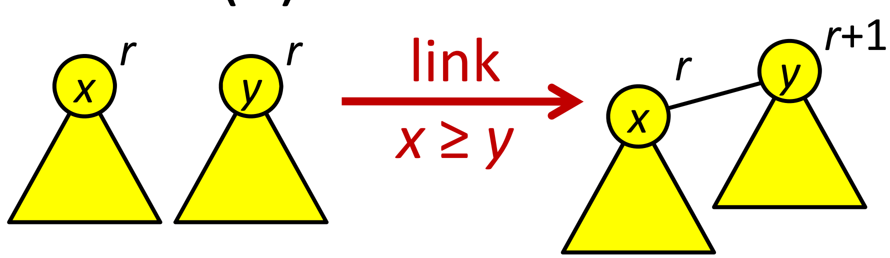
- Binomial **tree**
  - each node stores a (*k*,*p*) and satisfies **heap order** with respect to priorities
  - all nodes have a **rank** *r* (leaf = rank 0, a rank *r* node has exactly one child of each of the ranks 0..*r*-1)
- Binomial **queue**
  - forest of binomial trees with roots stored in a list with strictly increasing root ranks

Problem

Implement binomial queue operations to achieve the ideal times in the **amortized** sense

Hints

1) Two rank *i* trees can be linked to create a rank *i*+1 tree in $O(1)$ time



2) Potential Φ = max rank + #roots

Dijkstra's Algorithm

(Single source shortest path problem)

```
Algorithm Dijkstra(V, E, w, s)
    Q := MakeQueue
    dist[s] := 0
    Insert(Q, s, 0)
    for v ∈ V \ { s } do
        dist[v] := +∞
        Insert(Q, v, +∞)
    while Q ≠ ∅ do
        v := DeleteMin(Q)
        foreach u : (v, u) ∈ E do
            if u ∈ Q and dist[v]+w(v, u) < dist[u] then
                dist[u] := dist[v]+w(v, u)
                DecreaseKey(u, dist[u])
```

n x **Insert** + ***n*** x **DeleteMin** + ***m*** x **DecreaseKey**

Binary heaps / Binomial queues : $O((n + m) \cdot \log n)$

Priority Bounds

	Binomial Queues [Vuillemin 78]	Fibonacci Heaps [Fredman, Tarjan 84]	Run-Relaxed Heaps [Driscoll, Gabow, Shrairman, Tarjan 88]	[Brodal 96]
Insert	**1**	**1**	**1**	**1**
Meld	**1**	**1**	**-**	**1**
Delete	**log *n***	**log *n***	**log *n***	**log *n***
DeleteMin	**log *n***	**log *n***	**log *n***	**log *n***
DecreaseKey	**log *n***	**1**	**1**	**1**

Amortized **Worst-case**

⇓

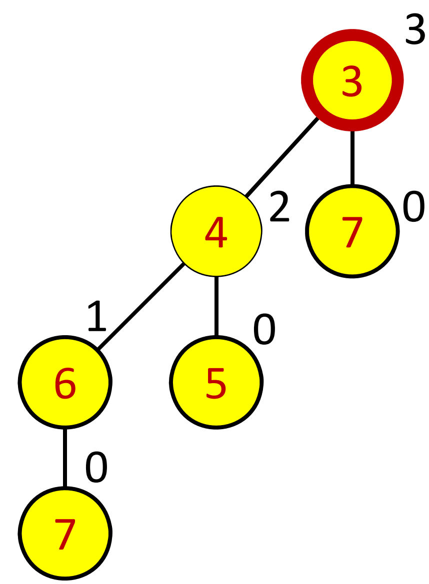
Dijkstra's Algorithm O(*m* + *n*·log *n*)

(and Minimum Spanning Tree O(*m*·log* *n*))

Empty, FindMin, Size, MakeQueue – O(1) worst-case time

Fibonacci Heaps

[Fredman, Tarjan, *Fibonacci Heaps and Their Use in Improved Network Algorithms,* Journal of the ACM, Volume 34(3), 596-615, 1987]



- F-tree
  - **heap order** with respect to priorities
  - all nodes have a **rank** $r \in$ {degree, degree + 1}
    (r = degree + 1 $\Leftrightarrow$ node is **marked** as having lost a child)
  - The ***i*'th child** of a node from the right has **rank ≥ *i* - 1**
- Fibonacci Heap
  - forest (list) of F-trees (trees can have equal rank)

Fibonnaci Heap Property

Thm Max rank of a node in an F-tree is O(log *n*)

Proof A rank *r* node has at least 2 children of rank $\geq r - 3$. By induction subtree size is at least $2^{\lfloor r/3 \rfloor}$ □

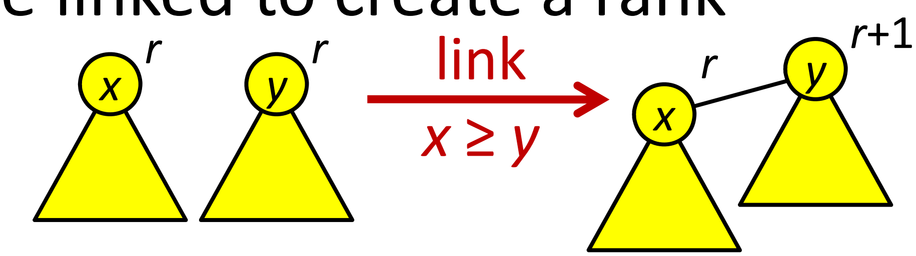
(in fact the size is at least φ^r , where $\varphi=(1+\sqrt{5})/2$)

Problem

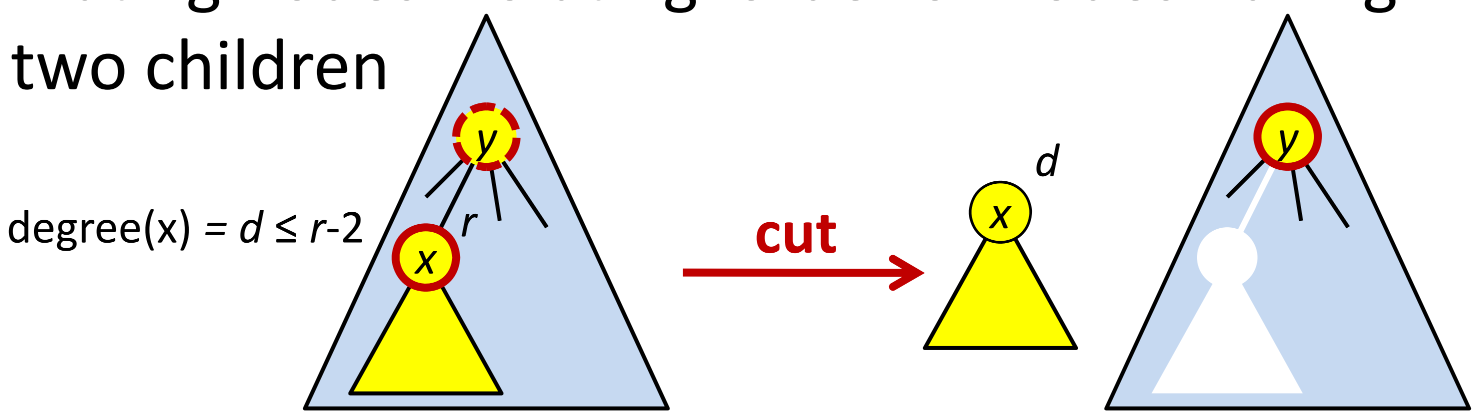
Implement Fibonacci Heap operations with **amortized** O(1) time for all operations, except O(log *n*) for deletions

Hints

1) Two rank *i* trees can be linked to create a rank *i*+1 tree in *O*(1) time



2) Eliminating nodes violating order or nodes having lost two children



3) Potential Φ = 2·marks + #roots

Implemenation of Fibonacci Heap Operations

FindMin	Maintain pointer to min root
Insert	Create new tree = new rank 0 node **+1**
Join	Concatenate two forests **unchanged**
Delete	DecreaseKey $-\infty$ + DeleteMin
DeleteMin	Remove min root **-1** + add children to forest **+O(log *n*)** + bucketsort roots by rank **only O(log *n*) not linked below** + link while two roots equal rank **-1 each**
DecreaseKey	Update priority + cut edge to parent **+3** + if parent now has $r - 2$ children, recursively cut parent edges **-1 each, +1 final cut**

***** = potential change

Worst-Case Operations

(without Join)

[Driscoll, Gabow, Shrairman, Tarjan, *Relaxed Heaps: An Alternative to Fibonacci Heaps with Applications to Parallel Computation,* Communications of the ACM, Volume 34(3), 596-615, 1987]

Basic ideas

- Require ≤ max-rank + 1 **trees** in forest (otherwise ∃rank r where two trees can be linked)
- Replace cutting in F-trees by having O(log n) nodes **violating heap order**
- **Transformation** replacing two rank *r* violations by one rank *r*+1 violation